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ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Keller–Osserman estimates and removability result for the anisotropic porous medium equation with gradient absorption term

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Abstract

We study "large" nonnegative solutions for a class of quasilinear equations model of which is

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1} u_{x_i}\right)_{x_i} + \sum_{i=1}^{n} \left|u_{x_i}\right|^{q_i} = 0.$$

We give a sufficient condition on the exponents m_i and q_i for the removability of isolated singularities.

KEYWORDS
anisotropic porous medium equation, Keller–Osserman a priori estimates, removability of isolated singularity

MSC (2010)
35B40, 35B45

1 | INTRODUCTION AND MAIN RESULTS

In this paper we study nonnegative solutions to the quasilinear parabolic equation in the divergent form. This class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs [4,17,38,40] and references therein)

$$u_t - divA(x,t,u,\nabla u) + g(x,t,\nabla u) = b(x,t,u,\nabla u), \quad (x,t) \in \Omega_T = \Omega \times (0,T), \tag{1.1}$$

where Ω is a bounded domain in $R^n, n \geq 2, 0 < T < +\infty, 0 \in \Omega$, satisfying the initial condition

$$u(x,0) = 0, \qquad x \in \Omega \setminus \{0\}. \tag{1.2}$$

Throughout the paper we suppose that the functions $A : \Omega \times R^1_+ \times R^1_+ \times R^n \to R^n$, $g, b : \Omega \times R^1_+ \times R^1_+ \times R^n \to R^1$ and such that $A(.,.,u,\xi)$, $g(.,.,\xi)$, $b(.,.,u,\xi)$ are Lebesgue measurable for all $u \in R^1_+, \xi \in R^n$, and $A(x,t,.,.)$, $g(x,t,.)$, $b(x,t,.,.)$ are continuous for almost all $(x,t) \in \Omega_T$, $A = (a_1, a_2, \dots, a_n)$. We also assume that the following structure conditions are satisfied

$$A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} u^{m_i-1}|\xi_i|^2,$$

$$|a_i(x,t,u,\xi)| \leq \nu_2 u^{m_i-1}|\xi_i|, \ i = \overline{1,n},$$

$$\nu_1 \sum_{i=1}^{n} |\xi_i|^{q_i} \le g(x,t,\xi) \le \nu_2 \sum_{i=1}^{n} |\xi_i|^{q_i},$$

$$|b(x,t,u,\xi)| \le \nu_2 u^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} u^{m_i-1} |\xi_i|^2 \right)^{\frac{1}{2}}, \tag{1.3}$$

where ν_1, ν_2 are positive constants and

$$1 - \frac{2}{n} < m_1 \le m_2 \le \dots \le m_n < m + \frac{2}{n}, \quad m = \frac{1}{n} \sum_{i=1}^{n} m_i, \tag{1.4}$$

$$\frac{2+nm}{1+n} \le q < 2, \quad \max_{0 \le i \le n} q_i < q\left(1 + \frac{1}{n}\right), \quad \frac{1}{q} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i}. \tag{1.5}$$

The qualitative behavior of solutions to quasilinear elliptic and parabolic equations near the point singularity was investigated by many authors stating from the seminal papers of Serrin [30,31] and Aronson, Serrin [3]. The question of the removability of isolated singularity for the Laplace equation with absorption was studied by Brezis and Veron [14]. Subsequently various extension of these results have been obtained by many authors. We refer to the monograph by Veron [39] for an account of these results.

A model example of an equation (1.1) in the nonanisotropic case ($m = m_1 = \dots = m_n, q = q_1 = \dots = q_n$) is

$$u_t - \triangle\left(|u|^{m-1} u\right) + |\nabla u|^q = 0,$$

existence and nonexistence of singular solutions of such type equations were considered in [1,2,5–8,10–12,16,19,20,24,28,29,32, 37]. Particularly, the removability of isolated singularity for solutions of these equations has been proved under the assumption $q \ge \frac{2+nm}{1+n}$.

During the last decade has been growing interest and substantial development in the qualitative theory of second order quasilinear elliptic and parabolic equations with nonstandart growth conditions. Some results of [15,18,21,23,25–27,33–36] we mention here. The basic prototypes of such equations are

$$\sum_{i=1}^{n} \left(\left| u_{x_i} \right|^{p_i-2} u_{x_i} \right)_{x_i} = 0, \tag{1.6}$$

$$u_t - \sum_{i=1}^{n} \left(\left| u_{x_i} \right|^{p_i-2} u_{x_i} \right)_{x_i} = 0, \tag{1.7}$$

$$u_t - \sum_{i=1}^{n} \left(|u|^{m_i-1} u_{x_i} \right)_{x_i} = 0. \tag{1.8}$$

The question of the removability of isolated singularity for anisotropic elliptic equations (1.6) with gradient absorption term was studied in [35].

The main goal of our paper is to establish the pointwise estimates of solutions to the problem (1.1), (1.2), depending on the relations between the exponents m_i and q_i which guarantee that point singularity is removable. The main difficulty lies in the fact that part of $m_i < 1$ (singular case), and another part of $m_i > 1$ (degenerate case). Particularly, we cover the case $m = 1$.

Before formulation the main results, let us remind the reader the definition of a weak solution to the problem (1.1), (1.2). We say that $u \in L^{\bar{q}}\left(0, T; W^{1,\bar{q}}(\Omega)\right)$ if $\iint\limits_{\Omega_T} |u|^q dx\,dt + \sum\limits_{i=1}^{n} \iint\limits_{\Omega_T} \left| u_{x_i} \right|^{q_i} dx\,dt < \infty$. Let $m^- = \min(1, m_1)$, we also say that $u \in V_m(\Omega_T)$ if $u \in C\left(0, T; L^{1+m^-}(\Omega)\right)$ and $\sum\limits_{i=1}^{n} \iint\limits_{\Omega_T} |u|^{m_i+m^--2} \left| u_{x_i} \right|^2 dx\,dt < \infty$. By a weak solution of the problem (1.1), (1.2) we mean the function $u(x,t) \ge 0$ satisfying the inclusion $u\psi \in V_m(\Omega_T) \cap L^{\bar{q}}\left(0, T; W^{1,\bar{q}}(\Omega)\right) \cap L^\infty(\Omega_T)$ and the integral identity

$$\int\limits_{\Omega} u(x,\tau) \psi^p \varphi \, dx + \int\limits_0^\tau \int\limits_{\Omega} \left(-u(\psi^p \phi)_t + A(x,t,u,\nabla u) \nabla(\psi^p \varphi) + g(x,t,\nabla u) \psi^p \varphi - b(x,t,u,\nabla u) \psi^p \varphi \right) dx\,dt = 0 \tag{1.9}$$

holds true for $p = \max\left(2 + m_n, \max_{1\le i\le n} q_i\right)$, for any $0 < \tau < T$, any testing function $\varphi \in W^{1,2}\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; \overset{o}{W}{}^{1,2}(\Omega)\right)$ and any $\psi \in C^1\left(\overline{\Omega}_T\right)$ vanishing in a neighborhood of $\{(0,0)\}$.

This guarantees that $\lim_{\tau\to 0} \int_\Omega u(x,\tau)\psi^p\varphi\, dx = 0$ and all the integrals in (1.9) are convergent.

Our main result reads as follows.

Theorem 1.1. *Let u be a weak nonnegative solution to the problem (1.1), (1.2). Let the conditions (1.3)–(1.5) be fulfilled, and assume also that if $q = \frac{2+nm}{1+n}$ then $q_i = \frac{2+nm}{1+n+\frac{n}{2}(m-m_i)}$, $i = \overline{1,n}$. Then the singularity at the point $x = 0$, $t = 0$ is removable, that is, the integral identity (1.9) holds for $\psi \equiv 1$.*

The proof of Theorem 1.1 is based on the new a priori estimates of "large" type solutions. In particular, we prove the following theorem, which is the Keller–Osserman type estimate of the solution to the problem (1.1), (1.2).

Theorem 1.2. *Let the conditions (1.3)–(1.5) be fulfilled. Then there exists a positive constant c depending only on $\nu_1, \nu_2, n, m_1, \dots, m_n, q_1, \dots, q_n$ such that the following inequality*

$$u(x,t) \le c\left(\sum_{i=1}^n |x_i|^{\frac{2}{(2-m)q+(q-2)m_i}} + t^{\frac{1}{q(1-m)+2(q-1)}}\right)^{q-2} \tag{1.10}$$

holds for $(x,t) \in \Omega_T \setminus \{(0,0)\}$.

The rest of the paper contains the proof of the above theorems.

2 | KELLER–OSSERMAN A PRIORI ESTIMATES, PROOF OF THEOREM 1.2

2.1 | Auxiliary propositions

The following lemmas will be used in the sequel. The first one is the well known embedding lemma (see [9]).

Lemma 2.1. *Let $\Omega \subset R^n$, $n \ge 2$, be a bounded domain, let $v \in \overset{o}{W}{}^{1,1}(\Omega)$ and let*

$$\sum_{i=1}^n \int_\Omega |v|^{\alpha_i}\left|v_{x_i}\right|^{p_i} dx < \infty, \quad \alpha_i \ge 0,\ p_i > 1.$$

If $1 < p < n$, then $v \in L^q(\Omega)$, $q = \frac{np}{n-p}\left(1 + \frac{1}{n}\sum_{i=1}^n \frac{\alpha_i}{p_i}\right)$, $\frac{1}{p} = \frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}$ and the following inequality

$$\|v\|_{L^q(\Omega)} \le \gamma \prod_{i=1}^n \left(\int_\Omega |v|^{\alpha_i}\left|v_{x_i}\right|^{p_i} dx\right)^{\frac{1}{np_i\left(1+\frac{1}{n}\sum_{i=1}^n \frac{\alpha_i}{p_i}\right)}}$$

holds, where the positive constant γ depends only on n, p_i, α_i, $i = \overline{1,n}$.

In what follows we will use the following lemma [17].

Lemma 2.2. *Let $\{y_j\}_{j\in N}$ be a sequence of nonnegative numbers such that for any $j = 0, 1, 2, \dots$ the inequality*

$$y_{j+1} \le C b^j y_j^{1+\varepsilon}$$

holds with positive ε, $C > 0$, $b > 1$. Then the following estimate

$$y_j \le C^{\frac{(1+\varepsilon)^j - 1}{\varepsilon}} b^{\frac{(1+\varepsilon)^j - 1}{\varepsilon^2} - \frac{j}{\varepsilon}} y_0^{(1+\varepsilon)^j}$$

is true. Particulary, if $y_0 \le C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}$, then $\lim_{j\to\infty} y_j = 0$.

2.2 | Integral estimates of solutions

For $r > 0$ set

$$D(r) = \left\{ (x,t) \in R^n \times R^1_+ \, : \, \sum_{i=1}^n |x_i|^{\frac{2(q-m)}{(2-m)q+(q-2)m_i}} + t^{\frac{q-m}{q(1-m)+2(q-1)}} < r \right\}.$$

Let R_0 be a sufficiently small fixed number such that $D(R_0) \subset \Omega_T$, and for $0 < r < R_0$ we define

$$M(r) := \operatorname{ess\,sup}\{u(x,t) \, : \, (x,t) \in D(R_0) \backslash D(r)\},$$

it follows from [21] that $M(r) < \infty$ for $0 < r < R_0$.

Fix a positive number $l \geq p+1$, in the sequel, γ stands for a constant which depends only on $l, v_1, v_2, n, m_1, \dots, m_n, q_1, \dots, q_n$ and R_0 which may vary from line to line. Let $0 < \rho < \frac{R_0}{2}$, $0 < \sigma < \frac{1}{2}$, $\frac{\rho}{2} \leq s(1-\sigma) \leq s \leq \frac{3}{2}\rho$, $u_{2\rho} = (u - M(2\rho))_+$, for $k > 0$ and $j = 0,1,2,\dots$ set $s_j = s(1-\sigma 2^{-j})$, $\bar{s}_{j+1} = \frac{1}{2}\left(s_j + s_{j+1}\right)$, $k_j = k(1-2^{-j})$, $Q_{s_j} = R^{n+1}_+ \setminus D\left(s_j\right)$, $A_{k_j,s_j} = \left\{(x,t) \in Q_{s_j} \, : \, u_{2\rho}(x,t) > k_j\right\}$. Let $\varsigma_j \in C^\infty\left(R^{n+1}_+\right)$, $0 \leq \varsigma_j \leq 1$, $\varsigma_j = 1$ in $Q_{s_{j+1}}$, $\varsigma_j = 0$ in $D\left(\bar{s}_j\right)$, and $\left|\frac{\partial \varsigma_j}{\partial t}\right| \leq \gamma\sigma^{-\gamma}2^{j\gamma}s^{-\frac{q(1-m)+2(q-1)}{q-m}}$, $\left|\frac{\partial \varsigma_j}{\partial x_i}\right| \leq \gamma\sigma^{-\gamma}2^{j\gamma}s^{-\frac{(2-m)q+(q-2)m_i}{2(q-m)}}$, $i = \overline{1,n}$.

Lemma 2.3. *Let the conditions of Theorem 1.2 be fulfilled. Then*

$$\sup_{0<t<T} \int_\Omega \left(u_{2\rho} - k_{j+1}\right)^2_+ \varsigma_j^l \, dx + \sum_{i=1}^n \iint_{\Omega_T} \left(u_{2\rho} - k_{j+1}\right)_+ \left|u_{x_i}\right|^{q_i} \varsigma_j^l \, dx\, dt \leq \gamma\sigma^{-\gamma}2^{j\gamma}\rho^{-\frac{2+q(1-m)}{q-m}} H(s,\rho)\left|A_{k_{j+1},\bar{s}_{j+1}}\right|, \tag{2.1}$$

where

$$H(s,\rho) = \left(\rho^{\frac{2-q}{q-m}} M(s(1-\sigma))\right)^2 + \sum_{i=1}^n \left(\rho^{\frac{2-q}{q-m}} M(s(1-\sigma))\right)^{m_i+1} + \rho^2\left(\rho^{\frac{2-q}{q-m}} M(s(1-\sigma))\right)^{m+1}.$$

Proof. Testing (1.9) by $\varphi = \left(u_{2\rho} - k_{j+1}\right)_+ \varsigma_j^{l-p}, \psi = \varsigma_j$, using conditions (1.3) and the Young inequality, we obtain (2.1), this proves the lemma. □

By Lemma 2.1 we obtain

$$\iint_{A_{k_{j+1},\bar{s}_{j+1}}} (u_{2\rho} - k_{j+1})^2_+ dx\,dt \leq \left(\iint_{A_{k_{j+1},\bar{s}_{j+1}}} (u_{2\rho} - k_{j+1})_+^{q+1+\frac{2q}{n}} dx\,dt\right)^{\frac{2}{q+1+\frac{2q}{n}}} \left|A_{k_{j+1},\bar{s}_{j+1}}\right|^{1-\frac{2}{q+1+\frac{2q}{n}}}$$

$$\leq \gamma\left(\sup_{0<t<T} \int_{A_{k_{j+1},\bar{s}_{j+1}}\times\{t\}} (u_{2\rho} - k_{j+1})^2_+ \varsigma_j^l \, dx\right)^{\frac{2q}{n(q+1+\frac{2q}{n})}}$$

$$\times\left(\prod_{i=1}^n\left(\iint_{A_{k_{j+1},\bar{s}_{j+1}}} (u_{2\rho} - k_{j+1})_+ \left|u_{x_i}\right|^{q_i} dx\,dt\right)^{\frac{1}{nq_i}}\right)^{\frac{2}{q+1+\frac{2q}{n}}} \left|A_{k_{j+1},\bar{s}_{j+1}}\right|^{1-\frac{2}{q+1+\frac{2q}{n}}},$$

from this, using the evident inequality $\left|A_{k_{j+1},\bar{s}_{j+1}}\right| \leq \gamma 2^{j\gamma}k^{-2} \iint_{A_{k_j,\bar{s}_j}} (u_{2\rho} - k_j)^2_+ dx\,dt$, by Lemma 2.3 we obtain

$$y_{j+1} = \iint_{A_{k_{j+1},\bar{s}_{j+1}}} (u_{2\rho} - k_{j+1})^2_+ dx\,dt$$

$$\leq \gamma\sigma^{-\gamma}2^{j\gamma}k^{-2\left(1+\frac{2q}{n(q+1+\frac{2q}{n})}\right)}\rho^{-\frac{2+q(1-m)}{q-m}\left(1+\frac{q}{n}\right)\frac{2}{q+1+\frac{2q}{n}}}(H(s,\rho))^{\left(1+\frac{q}{n}\right)\frac{2}{q+1+\frac{2q}{n}}} y_j^{1+\frac{2q}{n(q+1+\frac{2q}{n})}}, \quad j = 0,1,2\dots$$

By Lemma 2.2, $y_j \to 0$ as $j \to \infty$, provide k is chosen from

$$k^{4+n+\frac{n}{q}} = \rho^{-\frac{2+q(1-m)}{q-m}\left(1+\frac{n}{q}\right)} H^{1+\frac{n}{q}}(s,\rho) \iint\limits_{Q_{\bar{s}_0}} u_{2\rho}^2 \, dx \, dt. \tag{2.2}$$

Let us estimate the integral on the right-hand side of (2.2). Let $\varsigma \in C^\infty\left(R_+^{n+1}\right)$, $0 \le \varsigma \le 1$, $\varsigma = 1$ in $Q_{\bar{s}_0}$, $\varsigma = 0$ in $D(s(1-\sigma))$, and $\left|\frac{\partial \varsigma}{\partial t}\right| \le \gamma \sigma^{-\gamma} s^{-\frac{q(1-m)+2(q-1)}{q-m}}$, $\left|\frac{\partial \varsigma}{\partial x_i}\right| \le \gamma \sigma^{-\gamma} s^{-\frac{(2-m)q+(q-2)m_i}{2(q-m)}}$, $i = \overline{1,n}$.

Testing (1.9) by $\varphi = u_{2\rho}\varsigma^{l-p}$, $\psi = \varsigma$, and using the inequality $D(s) \le \gamma s^{n+\frac{q(1-m)+2(q-1)}{q-m}}$, similarly to (2.1) we have

$$\iint\limits_{Q_{\bar{s}_0}} u_{2\rho}^2 \, dx \, dt \le \gamma s^{\frac{q(1-m)+2(q-1)}{q-m}} \sup_{0<t<T} \int\limits_{Q_{\bar{s}_0}\times\{t\}} u_{2\rho}^2 \varsigma^l dx \le \gamma \sigma^{-\gamma} \rho^{n+2\frac{q(1-m)+2(q-1)}{q-m} - \frac{2+q(1-m)}{q-m}} H(s,\rho).$$

Hence (2.2) implies

$$(M(s) - M(2\rho))^{4+n+\frac{n}{q}} \le \gamma \sigma^{-\gamma} \rho^{n+2\frac{q(1-m)+2(q-1)}{q-m} - \frac{2+q(1-m)}{q-m}\left(2+\frac{n}{q}\right)} H^{2+\frac{n}{q}}(s,\rho).$$

If $\varepsilon \in (0,1)$ then by (1.4), (1.5) and the Young inequality from the previous inequality we obtain

$$M(s) \le \varepsilon M(s(1-\sigma)) + M(2\rho) + \gamma \sigma^{-\gamma} \varepsilon^{-\gamma} \rho^{\frac{q-2}{q-m}}. \tag{2.3}$$

For $j = 0,1,2,\dots$ define the sequences $\{\sigma_j\}$, $\{s_j\}$, $\{M_j\}$ by $\sigma_j := \frac{1}{2+2^j}$, $s_j := \frac{\rho}{2}\left(1+2^{-j+1}\right)$, $M_j = M\left(s_j\right)$, from (2.3) we arrive at recursive inequalities.

$$M_j \le \varepsilon M_{j+1} + M(2\rho) + \gamma 2^{j\gamma} \varepsilon^{-\gamma} \rho^{\frac{q-2}{q-m}}, \quad j = 0,1,2\dots$$

From this, by iteration

$$M_0 \le \varepsilon^j M_{j+1} + M(2\rho) \sum_{i=0}^{j} \varepsilon^i + \gamma \varepsilon^{-\gamma} \rho^{\frac{q-2}{q-m}} \sum_{i=0}^{j} (\varepsilon 2^\gamma)^i,$$

for every $j \ge 1$. We choose $\varepsilon \le 2^{-\gamma-1}\varepsilon_0$ such that the second sum on the right-hand side can be majorized by a convergent series and let $j \to \infty$ to obtain

$$M(\rho) \le M_0 \le (1-\varepsilon)^{-1} M(2\rho) + \gamma \varepsilon^{-\gamma} \rho^{\frac{q-2}{q-m}}.$$

For $j = 0,1,2,\dots$ define the sequence $\rho_j := 2^{-j} R_0$, then the previous inequality can be written in the form

$$M\left(\rho_j\right) \le (1-\varepsilon)^{-1} M\left(\rho_{j-1}\right) + \gamma \varepsilon^{-\gamma} \rho_j^{\frac{q-2}{q-m}}, \quad j = 1,2\dots$$

From this, by iteration

$$M\left(\rho_j\right) \le (1-\varepsilon)^{-j} M(R_0) + \gamma \varepsilon^{-\gamma} \rho_j^{\frac{q-2}{q-m}} \sum_{i=0}^{j} \left(\frac{2^{-\frac{2-q}{q-m}}}{1-\varepsilon}\right)^i,$$

choosing ε from the condition $\varepsilon < 1 - 2^{-\frac{q-2}{q-m}}$ so that the sum on the right-hand side of the previous inequality can be majorized by a convergent series to obtain

$$M\left(\rho_j\right) \le \gamma \rho_j^{\frac{q-2}{q-m}} \left(R_0^{\frac{2-q}{q-m}} M(R_0) + 1\right), \quad j = 1,2,\dots,$$

which proves Theorem 1.2. □

3 | PROOF OF THEOREM 1.1

3.1 | Integral estimates for the gradient of solutions

For $0 \le \lambda < n$ let us define the following numbers

$$\kappa(\lambda) = 2 + (n-\lambda)(m-1), \quad a_i(\lambda) = 1 + \frac{n-\lambda}{2}(m-m_i), \ i = \overline{1,n}, \tag{3.1}$$

and set

$$\rho_\lambda(x,t) = \left\{ \left(\sum_{i=1}^n |x_i|^{\frac{a_1(\lambda)}{a_i(\lambda)}} \right)^{\frac{\kappa(\lambda)}{a_1(\lambda)}} + t \right\}^{\frac{1}{\kappa}}, \quad G_\lambda(r) = \left\{ (x,t) \in R^n \times R^1_+ : \rho_\lambda(x,t) < r \right\}.$$

Assume that $G_\lambda(R_0) \subset \Omega_T$ and for $0 < r < R_0$ we set $M_r(\lambda) = \sup\{|u(x,t)| : (x,t) \in G_\lambda(R_0) \setminus G_\lambda(r)\}$, $E(r,\lambda) = \{(x,t) \in \Omega_T : u(x,t) > M_r(\lambda)\}$, $u_r(x,t,\lambda) = (u(x,t) - M_r(\lambda))_+$ and consider the function $\psi_r(x,t) = \eta_r(\rho_\lambda(x,t))$, where $\eta_r : R^1 \to R^1$ is a function taking the following values: $\eta_r(s) = 0$ if $s \le r$, $\eta_r(s) = 1$ if $s \ge R(r)$, $\eta_r(s) = \left[(1-\delta)\ln\ln\frac{1}{r}\right]^{-1}\left(\ln\ln\frac{1}{r} - \ln\ln\frac{1}{s}\right)$ if $r \le s \le R(r)$, here δ is a number from the interval $(0,1)$ specified in what follows and $R(r)$ defined by the equality

$$\ln\frac{1}{R(r)} = \ln^\delta \frac{1}{r}. \tag{3.2}$$

Note that by the evident equalities

$$\frac{q-2}{q(1-m)+2(q-1)} = -\frac{n-\lambda}{\kappa(\lambda)}, \quad \frac{2(q-2)}{(2-m)q+(q-2)m_i} = -\frac{n-\lambda}{a_i(\lambda)}, \quad i = \overline{1,n},$$

with $\lambda \ge 0$ defined by

$$\lambda = \frac{q(1+n) - 2 - nm}{q-m} \ge 0,$$

the Keller–Osserman estimate (1.10) yields

$$M_\rho(\lambda) \le \gamma \rho^{\lambda-n}, \quad \rho > 0. \tag{3.3}$$

To simplify the following calculations we will write M_r, $E(r)$ and $u_r(x,t)$ instead of $M_r(\lambda)$, $E(r,\lambda)$ and $u_r(x,t,\lambda)$.

In the case $\lambda = 0$, i.e. $q = \frac{2+nm}{1+n}$ and $q_i = \frac{2+nm}{1+n+\frac{n}{2}(m-m_i)}$, $i = \overline{i,n}$, fix $\varepsilon_1, \varepsilon_2 \in (0,1)$ by the conditions

$$\frac{1}{q_n} < \varepsilon_1 < \left(\frac{1}{q} - \frac{1}{n}\right)\frac{q}{q_n - q}, \quad \frac{1}{n} + \frac{\varepsilon_1 q_n - 1}{q} < \varepsilon_2 < \varepsilon_1,$$

and let

$$0 < \frac{\varepsilon_1 q_n - 1}{q\left(\varepsilon_2 - \frac{1}{n}\right)} < \delta < 1. \tag{3.4}$$

Consider the functions $F_1(r,\lambda)$, $F_2(r,\lambda)$, $F_3(r,\lambda)$ defined by the following equalities

$$F_1(r,\lambda) = \begin{cases} R^\lambda(r), & \text{if } \lambda > 0, \\ \left[\ln\frac{1}{R(r)}\right]^{2-q_1'}, & \text{if } \lambda = 0, \end{cases}$$

$$F_2(r,\lambda)=\begin{cases} R^{\lambda}(r), & \text{if } \lambda>0,\\ \left[\ln\frac{1}{r}\right]^{1-(\varepsilon_1-\varepsilon_2)q'}, & \text{if } \lambda=0,\ (\varepsilon_1-\varepsilon_2)q'<1,\\ \ln\ln\frac{1}{r}, & \text{if } \lambda=0,\ (\varepsilon_1-\varepsilon_2)q'=1,\\ \left[\ln\frac{1}{R(r)}\right]^{1-(\varepsilon_1-\varepsilon_2)q'}, & \text{if } \lambda=0,\ (\varepsilon_1-\varepsilon_2)q'>1. \end{cases}$$

$$F_3(r,\lambda)=\begin{cases} R^{\lambda}(r), & \text{if } \lambda>0,\\ \left[\ln\frac{1}{r}\right]^{1-(1-\delta_1)q_1-q\delta\left(\varepsilon_2-\frac{1}{n}\right)}, & \text{if } \lambda=0,\ (1-\varepsilon_1)q_1<1,\\ \left[\ln\frac{1}{r}\right]^{-q\delta\left(\varepsilon_2-\frac{1}{n}\right)}, & \text{if } \lambda=0,\ (1-\varepsilon_1)q_1=1,\\ \left[\ln\frac{1}{R(r)}\right]^{1-(1-\varepsilon_1)q_1-q\left(\varepsilon_2-\frac{1}{n}\right)}, & \text{if } \lambda=0,\ (1-\varepsilon_1)q_1>1. \end{cases}$$

Here $q'=\frac{q}{q-1}$, $q_i'=\frac{q_i}{q_i-1}$, $i=\overline{1,n}$.

Lemma 3.1. *Let the assumptions of Theorem 1.1 be fulfilled. Then the following estimate holds:*

$$I_0=\sup_{0<t<T}\int\limits_{E\left(\frac{\rho}{2}\right)\times\{t\}}\int\limits_{M_{\frac{\rho}{2}}}^{u}\ln\frac{s}{M_{\frac{\rho}{2}}}ds\,\psi_r^l\,dx+\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}u^{m_i-2}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt+\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}\ln\frac{u}{M_{\frac{\rho}{2}}}\left|u_{x_i}\right|^{q_i}\psi_r^l\,dx\,dt$$
$$\le\gamma(F_1(r,\lambda)+F_2(r,\lambda)+F_3(r,\lambda))+\gamma\rho^2\ln^2\frac{1}{\rho}. \tag{3.5}$$

Proof. Testing (1.9) by $\varphi=\ln_+\frac{u}{M_{\frac{\rho}{2}}}\psi_r^{l-p}$, $\psi=\psi_r$, using conditions (1.3) and the Young inequality we obtain

$$I_o\le\gamma\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}\int\limits_{M_{\frac{\rho}{2}}}^{u}\ln\frac{s}{M_{\frac{\rho}{2}}}ds\left|\frac{\partial\psi_r}{\partial t}\right|\psi_r^{l-1}dx\,dt+\gamma\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}\ln\frac{u}{M_{\frac{\rho}{2}}}u^{m_i-1}\left|u_{x_i}\right|\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{l-1}dx\,dt$$
$$+\gamma\iint\limits_{E\left(\frac{\rho}{2}\right)}u^m\ln^2\frac{u}{M_{\frac{\rho}{2}}}\psi_r^l dx\,dt=I_1+I_2+I_3. \tag{3.6}$$

By (3.3) the last term on the right-hand side of (3.6) is estimated as follows

$$I_3\le\gamma\rho^2\ln^2\frac{1}{\rho}. \tag{3.7}$$

If $\lambda>0$, then by Young's inequality and by (3.3) we have

$$I_1+I_2-\frac{1}{4}\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}u^{m_i-2}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt$$
$$\le\gamma\iint\limits_{E\left(\frac{\rho}{2}\right)}(u-M_{\frac{\rho}{2}})\ln\frac{u}{M_{\frac{\rho}{2}}}\left|\frac{\partial\psi_r}{\partial t}\right|\psi_r^{l-1}dx\,dt+\gamma\sum_{i=1}^n\iint\limits_{E\left(\frac{\rho}{2}\right)}u^{m_i}\ln^2\frac{u}{M_{\frac{\rho}{2}}}\left|\frac{\partial\psi_r}{\partial x_i}\right|^2\psi_r^{l-2}dx\,dt$$
$$\le\gamma\iint\limits_{G_\lambda(R(r))}\rho_\lambda^{\lambda-n-\kappa(\lambda)}dx\,dt+\gamma\sum_{i=1}^n\iint\limits_{G_\lambda(R(r))}\rho_\lambda^{(\lambda-n)m_i-2a_i(\lambda)}dx\,dt$$
$$\le\gamma\int\limits_0^{R(r)}s^{\lambda-1}ds\le\gamma R^\lambda(r). \tag{3.8}$$

If $\lambda = 0$, then by the Young inequality we have

$$I_2 - \frac{1}{4}\sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} \ln\frac{u}{M_{\frac{\rho}{2}}}\left|u_{x_i}\right|^{q_i}\psi_r^l\,dx\,dt \le \gamma\sum_{i=1}^{n}\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{(m_i-1)q_i'}\ln\frac{u}{M_{\frac{\rho}{2}}}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{q_i'}\psi_r^{l-q_i'}\,dx\,dt$$
$$\le \gamma\sum_{i=1}^{n}\iint\limits_{G_\lambda(R(r))\setminus G_\lambda(r)} \rho_0^{-(n(m_i-1)+2a_i(0))q_i'}\left[\ln\frac{1}{\rho_0}\right]^{1-q_i'}\,dx\,dt$$
$$\le \gamma\sum_{i=1}^{n}\int_r^{R(r)}\left[\ln\frac{1}{s}\right]^{1-q_i'}\frac{ds}{s} \le \gamma F_1(r,\lambda). \tag{3.9}$$

Set

$$v = \left(\int_{M_{\frac{\rho}{2}}}^{u}\ln^a\frac{s}{M_{\frac{\rho}{2}}}\,ds\right)_+\psi_r^{l-1}, \quad \rho(x) = \left(\sum_{i=1}^{n}|x_i|^{\frac{a_1(0)}{a_i(0)}}\right)^{\frac{1}{a_1(0)}},$$

then by the Young inequality we obtain

$$I_1 \le \gamma\iint\limits_{E\left(\frac{\rho}{2}\right)}\left[\ln\frac{u}{M_{\frac{\rho}{2}}}\right]^{(1-\varepsilon_1)q'}\left[\ln\frac{1}{\rho(x)}\right]^{\varepsilon_2 q'}\rho^{q'}(x)\left|\frac{\partial\psi_r}{\partial t}\right|^{q'}dx\,dt + \gamma\iint\limits_{E\left(\frac{\rho}{2}\right)} v^q\rho^{-q}(x)\left[\ln\frac{1}{\rho(x)}\right]^{\varepsilon_2 q}dx\,dt = I_4 + I_5. \tag{3.10}$$

By (3.3) we estimate the first term on the right-hand side of the previous inequality as follows

$$I_4 \le \gamma\int_r^{R(r)}\left[\ln\frac{1}{s}\right]^{-(\varepsilon_1-\varepsilon_2)q'}\frac{ds}{s} \le \gamma F_2(r,\lambda), \tag{3.11}$$

here we also used the evident equality $q' - \kappa(0)q' = -2 - nm$.

The second term on the right-hand side of (3.10) we estimate, using the Hölder inequality and Lemma 2.1 with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, we have

$$I_5 \le \gamma\int_0^T\left(\int\limits_{G_0(R(r))}\rho^{-n}(x)\left[\ln\frac{1}{\rho(x)}\right]^{-\varepsilon_2 n}dx\right)^{\frac{q}{n}}\left(\int\limits_{E\left(\frac{\rho}{2}\right)\times\{t\}} v^{\frac{nq}{n-q}}\,dx\right)^{\frac{n-q}{n}}dt$$
$$\le \gamma\left[\ln\frac{1}{R(r)}\right]^{-q\left(\varepsilon_2-\frac{1}{n}\right)}\sum_{i=1}^{n}\iint\limits_{E\left(\frac{\rho}{2}\right)}|v_{x_i}|^{q_i}\,dx\,dt$$
$$\le \gamma\left[\ln\frac{1}{R(r)}\right]^{-q\left(\varepsilon_2-\frac{1}{n}\right)}\sum_{i=1}^{n}\iint\limits_{E\left(\frac{\rho}{2}\right)}\ln^{\varepsilon_1 q_i}\frac{u}{M_{\frac{\rho}{2}}}\left|u_{x_i}\right|^{q_i}\psi_r^{(l-1)q_i}\,dx\,dt$$
$$+\gamma\left[\ln\frac{1}{R(r)}\right]^{-q\left(\varepsilon_2-\frac{1}{n}\right)}\sum_{i=1}^{n}\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{q_i}\left[\ln\frac{u}{M_{\frac{\rho}{2}}}\right]^{\varepsilon_1 q_i}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{q_i}\psi_r^{(l-2)q_i}\,dx\,dt.$$

By our assumptions $\varepsilon_1 q_i > 1, i = \overline{1,n}$, hence choosing l sufficiently large, using (3.3), from the previous we obtain

$$I_5 \le \gamma \left[\ln \frac{1}{R(r)}\right]^{-q\left(\varepsilon_2-\frac{1}{n}\right)} \left[\ln \frac{1}{r}\right]^{\varepsilon_1 q_n-1} \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} \ln \frac{u}{M_{\frac{\rho}{2}}} \left|u_{x_i}\right|^{q_i} \psi_r^l \, dx\, dt$$

$$+\gamma \left[\ln \frac{1}{R(r)}\right]^{-q\left(\varepsilon_2-\frac{1}{n}\right)} \sum_{i=1}^{n} \int\limits_{r}^{R(r)} \left[\ln \frac{1}{s}\right]^{(\varepsilon_1-1)q_i} \frac{ds}{s},$$

by our choice $\varepsilon_1 q_n - 1 - \delta q\left(\varepsilon_2 - \frac{1}{n}\right) < 0$, hence assuming that R_0 is small enough, satisfying the condition

$$\left[\ln \frac{1}{R_0}\right]^{\varepsilon_1 q_n-1-\delta q\left(\varepsilon_2-\frac{1}{n}\right)} \le \frac{1}{4\gamma},$$

we obtain

$$I_5 \le \frac{1}{4} \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} \ln \frac{u}{M_{\frac{\rho}{2}}} \left|u_{x_i}\right|^{q_i} \psi_r^l \, dx\, dt + \gamma F_3(r,\lambda). \tag{3.12}$$

Collecting estimates (3.6)–(3.12) we arrive at the required (3.5). □

In the case $\lambda = 0$, i. e. $q = \frac{2+nm}{1+n}$ and $q_i = \frac{2+nm}{1+n+\frac{n}{2}(m-m_i)}$, $i = \overline{1,n}$, fix numbers $\varepsilon_3, \varepsilon_4, \varepsilon_5$ by the conditions $\varepsilon_3 \in (0,1)$, $\frac{1-\varepsilon_3}{q_1} < \varepsilon_4 < 1$, $\frac{1-\varepsilon_3}{n} < \varepsilon_5 < \frac{1-\varepsilon_3}{q}$, and let

$$F_4(r,\lambda) = \begin{cases} R^{\lambda}(r), & \text{if } \lambda > 0, \\ \ln^{-1} \frac{1}{R(r)}, & \text{if } \lambda = 0, \end{cases}$$

$$F_5(r,\lambda) = \begin{cases} R^{\lambda}(r), & \text{if } \lambda > 0, \\ \left[\ln \frac{1}{R(r)}\right]^{1-(1-\varepsilon_5)\bar q'}, & \text{if } \lambda = 0, \end{cases}$$

$$F_6(r,\lambda) = \begin{cases} R^{\lambda}(r), & \text{if } \lambda > 0, \\ \left[\ln \frac{1}{R(r)}\right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)} (R(r))^{\varepsilon_3 n q_1}, & \text{if } \lambda = 0. \end{cases}$$

Here $\bar q = \frac{q}{1-\varepsilon_3}$, $\bar q' = \frac{\bar q}{\bar q - 1}$.

For $1 < \theta < 1 + \frac{2}{n}$ set $\Phi_\rho(u) = \left((M_{\frac{\rho}{2}} - M_{2\rho})^{1-\theta} - u_{2\rho}^{1-\theta}\right)_+$.

Lemma 3.2. *Under the conditions of Theorem 1.1 the following inequality*

$$\sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u_{2\rho}^{-\theta} u^{m_i-1} \left|u_{x_i}\right|^2 \psi_r^l \, dx\, dt + \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} \Phi_\rho(u) \left|u_{x_i}\right|^{q_i} \psi_r^l \, dx\, dt \le \gamma H_1(r,\rho,\lambda) \tag{3.13}$$

holds, where

$$H_1(r,\rho,\lambda) = \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} \left(F_1(r,\lambda) + F_2(r,\lambda) + F_3(r,\lambda) + \rho^2 \ln^2 \frac{1}{\rho}\right)^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda)$$

$$+ \left(\sum_{i=1}^{n} \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1}\right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} F_5(r,\lambda)$$

$$+ \sum_{i=1}^{n} \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3}} \left(\sum_{i=1}^{n} \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3}-1}\right)^{-1} F_6(r,\lambda) + \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{2(1-\theta)} \rho^{2+n-\theta n+\lambda\theta}.$$

Proof. Testing (1.9) by $\varphi = \Phi_\rho(u)\psi_r^{l-p}$, $\psi = \psi_r$ and using conditions (1.3) we get

$$\sup_{0<t<T} \int\limits_{E_{\frac{\rho}{2}}\times\{t\}} \int\limits_{M_{\frac{\rho}{2}}}^{u} \Phi_\rho(s)\,ds\,\psi_r^l\,dx + \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u_{2\rho}^{-\theta} u^{m_i-1} \left|u_{x_i}\right|^2 \psi_r^l\,dx\,dt + \sum_{i=1}^{n} \iint\limits_{E_{\frac{\rho}{2}}} \Phi_\rho(u) \left|u_{x_i}\right|^{q_i} \psi_r^l\,dx\,dt$$

$$\le \gamma \iint\limits_{E\left(\frac{\rho}{2}\right)} \int\limits_{M_{\frac{\rho}{2}}}^{u} \Phi_\rho(s)\,ds \left|\frac{\partial\psi_r}{\partial t}\right| \psi_r^{l-1} dx\,dt + \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-1}\left|u_{x_i}\right| \left|\frac{\partial\psi_r}{\partial x_i}\right| \psi_r^{l-1} dx\,dt$$

$$+ \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{\frac{m-1}{2}} \left(\sum_{i=1}^{n} u^{m_i-1}\left|u_{x_i}\right|^2\right)^{\frac{1}{2}} \psi_r^l\,dx\,dt$$

$$= I_6 + \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} (I_7 + I_8). \tag{3.14}$$

By (3.3) the Hölder inequality and Lemma 3.1 we have

$$I_7 \le \sum_{i=1}^{n} \left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-2}\left|u_{x_i}\right|^2 \psi_r^l\,dx\,dt \right)^{\frac{1}{2}} \left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i} \left|\frac{\partial\psi_r}{\partial x_i}\right|^2 dx\,dt \right)^{\frac{1}{2}}$$

$$\le \gamma\left(F_1(r,\lambda) + F_2(r,\lambda) + F_3(r,\lambda) + \rho^2 \ln^2 \frac{1}{\rho} \right)^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda). \tag{3.15}$$

By our choice of θ we have

$$\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} I_8 - \frac{1}{4\gamma} \sum_{i=1}^{n} \iint\limits_{E\left(\frac{\rho}{2}\right)} u_{2\rho}^{-\theta} u^{m_i-1} \left|u_{x_i}\right|^2 \psi_r^l\,dx\,dt$$

$$\le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{2(1-\theta)} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{\theta+m-1}\psi_r^l\,dx\,dt$$

$$\le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{2(1-\theta)} \iint\limits_{G_\lambda(2\rho)\setminus G_\lambda(r)} \rho_\lambda^{-(n-\lambda)(\theta+m-1)} dx\,dt$$

$$\le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{2(1-\theta)} \int\limits_{r}^{2\rho} s^{-(n-\lambda)(\theta+m-1)+2+(n-\lambda)(m-1)+n} \frac{ds}{s}$$

$$\le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{2(1-\theta)} \rho^{2+n-\theta n+\lambda\theta}. \tag{3.16}$$

If $\lambda > 0$ then by (3.3)

$$I_6 \le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} \iint\limits_{G_\lambda(R(r))\setminus G_\lambda(r)} \rho_\lambda^{\lambda-n-\kappa(\lambda)} \ln^{-1}\frac{1}{\rho_\lambda} dx\,dt \le \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{1-\theta} R^\lambda(r). \tag{3.17}$$

If $\lambda = 0$, set $w = \left(\int\limits_{M_{\frac{\rho}{2}}}^{u} \Phi_\rho^{\varepsilon_4}(s) s^{-\varepsilon_3}\, ds \right)_+ \psi_r^{l-1}$, then by the Young inequality we obtain

$$I_6 \le \gamma \left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1} \right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{\varepsilon_3 \bar q'} (\rho(x))^{(1-\varepsilon_3)\bar q'} \left[\ln \frac{1}{\rho(x)} \right]^{\varepsilon_5 \bar q'} \left| \frac{\partial \psi_r}{\partial t} \right|^{\bar q'} dx\, dt$$

$$+ \gamma \left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1} \right)^{-1} \iint\limits_{E\left(\frac{\rho}{2}\right)} w^{\bar q} \rho^{-q}(x) \left[\ln \frac{1}{\rho(x)} \right]^{-\varepsilon_5 \bar q} dx\, dt = I_9 + I_{10}. \tag{3.18}$$

By (3.3), the first term on the right-hand side of (3.18), we estimate as follows

$$I_9 \le \gamma \left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1} \right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} \int\limits_r^{R(r)} \left[\ln \frac{1}{s} \right]^{-(1-\varepsilon_5)\bar q'} \frac{ds}{s}$$

$$\le \gamma \left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1} \right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} F_5(r, \lambda), \tag{3.19}$$

here we also use the evident equality $(n\varepsilon_3 - 1 + \varepsilon_3 + \kappa(0))\bar q' = 2 + nm$.

The second term on the right-hand side of (3.18) we estimate, using the Höldeer inequality and Lemma 2.1 with $\alpha_i = \frac{\varepsilon_3}{1-\varepsilon_3} q_i$, $i = \overline{1, n}$, and choosing l sufficiently large, we have

$$I_{10} \sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4}{1-\varepsilon_3} q_i - 1} \le \gamma \int\limits_0^T \left(\int\limits_{G_0(R(r)) \times \{t\}} \rho^{-n}(x) \left[\ln \frac{1}{\rho(x)} \right]^{-\frac{\varepsilon_5 n}{1-\varepsilon_3}} dx \right)^{\frac{q}{n}} \left(\int\limits_{E\left(\frac{\rho}{2}\right) \times \{t\}} w^{\frac{n\bar q}{n-q}} dx \right)^{\frac{n-q}{n}} dt$$

$$\le \gamma \left[\ln \frac{1}{R(r)} \right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3} - \frac{1}{n}\right)} \sum_{i=1}^n \iint\limits_{E\left(\frac{\rho}{2}\right)} (\Phi_\rho(u))^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3}} \left| u_{x_i} \right|^{q_i} \psi_r^l\, dx\, dt$$

$$+ \gamma \left[\ln \frac{1}{R(r)} \right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3} - \frac{1}{n}\right)} \sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3}} \iint\limits_{E\left(\frac{\rho}{2}\right)} u^{(1-\varepsilon_3) q_i} \left| \frac{\partial \psi_r}{\partial x_i} \right|^{q_i} dx\, dt.$$

From this by our choice of $\varepsilon_3, \varepsilon_4, \varepsilon_5$, using the equality $(n + a_i(0)) q_i = 2 + nm$, and choosing R_0 from the condition

$$\left[\ln \frac{1}{R_0} \right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3} - \frac{1}{n}\right)} \le \frac{1}{4\gamma},$$

we obtain

$$I_{10} - \frac{1}{4} \sum_{i=1}^n \iint\limits_{E\left(\frac{\rho}{2}\right)} \Phi_\rho(u) \left| u_{x_i} \right|^{q_i} \psi_r^l\, dx\, dt \le \gamma \sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3}} \left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho} \right)^{\frac{\varepsilon_4 q_i}{1-\varepsilon_3} - 1} \right)^{-1} F_6(r, \lambda). \tag{3.20}$$

Collecting estimates (3.14)–(3.20) we arrive at the required (3.13). □

Define a function $u^{(\rho)}(x,t)$ and a set $E\left(\frac{\rho}{2},2\rho\right)$ as follows $u^{(\rho)} = \min\left(M_{\frac{\rho}{2}} - M_{2\rho}, u_{2\rho}\right)$, $E\left(\frac{\rho}{2},2\rho\right) = E(2\rho) \cap \left\{u < M_{\frac{\rho}{2}}\right\}$.

Lemma 3.3. *Under the assumptions of Theorem 1.1 the next inequality holds:*

$$\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{E(\frac{\rho}{2},2\rho)\times\{t\}} u_{2\rho}^2\,dx + \sum_{i=1}^n \iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m_i-1}\left|u_{x_i}\right|^2 dx\,dt + \sum_{i=1}^n \iint\limits_{E(2\rho)} u^{(\rho)}\left|u_{x_i}\right|^{q_i} dx\,dt \le \gamma\rho^{2-n+2\lambda}. \tag{3.21}$$

Proof. Testing (1.9) by $\varphi = u^{(\rho)}\,\psi_r^{l-p}$, $\psi = \psi_r$, using (1.3) we get

$$\begin{aligned}
&\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{E(\frac{\rho}{2},2\rho)\times\{t\}} u_{2\rho}^2\psi_r^l\,dx + \sum_{i=1}^n \iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m_i-1}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt + \sum_{i=1}^n \iint\limits_{E(2\rho)} u^{(\rho)}\left|u_{x_i}\right|^{q_i}\psi_r^l\,dx\,dt \\
&\le \gamma \iint\limits_{E(2\rho)} u^{(\rho)}u_{2\rho}\left|\frac{\partial\psi_r}{\partial t}\right|\psi_r^{l-1}dx\,dt + \gamma\sum_{i=1}^n \iint\limits_{E(2\rho)} u^{(\rho)}u^{m_i-1}\left|u_{x_i}\right|\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{l-1}dx\,dt \\
&+\gamma \iint\limits_{E(2\rho)} u^{\frac{m-1}{2}}u^{(\rho)}\left(\sum_{i=1}^n u^{m_i-1}\left|u_{x_i}\right|^2\right)^{\frac{1}{2}}\psi_r^l\,dx\,dt = I_{11}+I_{12}+I_{13}.
\end{aligned} \tag{3.22}$$

By the Hölder and Young inequalities and Lemmas 3.1, 3.2 we obtain

$$\begin{aligned}
&I_{12}+I_{13} - \frac{1}{4}\iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m_i-1}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt \\
&\le \gamma\sum_{i=1}^n \iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m_i+1}\left|\frac{\partial\psi_r}{\partial x_i}\right|^2\psi_r^{l-2}dx\,dt + \gamma\iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m+1}\psi_r^l\,dx\,dt \\
&+\gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)\sum_{i=1}^n\left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i-2}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt\right)^{\frac{1}{2}}\left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{m_i}\left|\frac{\partial\psi_r}{\partial x_i}\right|^2dx\,dt\right)^{\frac{1}{2}} \\
&+\gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)\sum_{i=1}^n\left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u_{2\rho}^{-\theta}u^{m_i-1}\left|u_{x_i}\right|^2\psi_r^l\,dx\,dt\right)^{\frac{1}{2}}\left(\iint\limits_{E\left(\frac{\rho}{2}\right)} u^{\theta+m-1}\psi_r^l\,dx\,dt\right)^{\frac{1}{2}} \\
&\le \gamma\sum_{i=1}^n M_{\frac{\rho}{2}}^{m_i+1}R^{nm_i}(r) + \gamma\rho^{2-n+2\lambda} \\
&+\gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)\left(F_1(r,\lambda)+F_2(r,\lambda)+F_3(r,\lambda)+\rho^2\ln\frac{1}{\rho}\right)^{\frac{1}{2}}F_4^{\frac{1}{2}}(r,\lambda) + \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right)H_1^{\frac{1}{2}}(r,\rho,\lambda)\rho^{\frac{2+n-\theta n+\lambda\theta}{2}}.
\end{aligned} \tag{3.23}$$

Let us estimate the first term on the right-hand side of (3.22). If $\lambda > 0$, then

$$I_{11} \leq \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right) \iint\limits_{G_\lambda(R(r))} \rho_\lambda^{\lambda-n-\kappa(\lambda)} \ln^{-1}\frac{1}{\rho_\lambda}\, dx\, dt \leq \gamma\left(M_{\frac{\rho}{2}} - M_{2\rho}\right) R^\lambda(r). \tag{3.24}$$

If $\lambda = 0$, set $\omega = (u^{(\rho)})^{\frac{1-\varepsilon_3}{q_1}} u^{1-\varepsilon_3}\psi_r^{l-1}$, where $\varepsilon_3 \in (0,1)$ is an arbitrary number. By the Young inequality we have

$$\begin{aligned}
I_{11} &\leq \gamma\left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} \iint\limits_{E(2\rho)} \left(u^{(\rho)}\right)^{\varepsilon_3\bar q'} u_{2\rho}^{\varepsilon_3\bar q'}(\rho(x))^{(1-\varepsilon_3)\bar q'}\left[\ln\frac{1}{\rho(x)}\right]^{\varepsilon_5\bar q'}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{\bar q'} dx\, dt\\
&\quad + \gamma\left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{-1} \iint\limits_{E(2\rho)} \omega^{\bar q}\rho^{-q}(x)\left[\ln\frac{1}{\rho(x)}\right]^{-\varepsilon_5\bar q} dx\, dt\\
&= I_{14} + I_{15}.
\end{aligned}$$

Here $\varepsilon_5, \bar q$ and $\bar q'$ were defined in Lemma 3.2. The first term on the right-hand side of the previous inequality was estimated in (3.19):

$$I_{14} \leq \gamma\left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\varepsilon_3\bar q'} F_5(r,\lambda). \tag{3.25}$$

The second term on the right-hand side of the previous inequality we estimate, using the Hölder inequality and Lemma 2.1 with $\alpha_i = \frac{\varepsilon_3}{1-\varepsilon_3} q_i$, $i = \overline{1,n}$ and choosing a sufficiently large l, we get

$$\begin{aligned}
I_{15}\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1} &\leq \gamma\left[\ln\frac{1}{R(r)}\right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)} \sum_{i=1}^n \iint\limits_{E(2\rho)} \left(u^{(\rho)}\right)^{\frac{q_i}{q_1}} \left|u_{x_i}\right|^{q_i}\psi_r^l\, dx\, dt\\
&\quad + \gamma\left(\ln\frac{1}{R(r)}\right)^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)} \sum_{i=1}^n \iint\limits_{E(2\rho)} \left(u^{(\rho)}\right)^{\frac{\varepsilon_3 q_i}{q_1}} u^{\varepsilon_3 q_i}\left|\frac{\partial\psi_r}{\partial x_i}\right|^{q_i} dx\, dt,
\end{aligned}$$

from this, choosing R_0 sufficiently small, so that $\left[\ln\frac{1}{R_0}\right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)} \leq \frac{1}{4\gamma}$, we obtain

$$\begin{aligned}
&I_{15} - \frac{1}{4}\sum_{i=1}^n \iint\limits_{E(2\rho)} u^{(\rho)}\left|u_{x_i}\right|^{q_i}\psi_r^l\, dx\, dt\\
&\leq \gamma\left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{-1} \sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{\varepsilon_3 q_i}{q_1}} \left[\ln\frac{1}{R(r)}\right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)-q_1} (R(r))^{n(1-\varepsilon_3)q_1}.
\end{aligned} \tag{3.26}$$

Combining estimates (3.22)–(3.26) we arrive at

$$\begin{aligned}
&\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{E(\frac{\rho}{2},2\rho)} u_{2\rho}^2\psi_r^l\, dx + \sum_{i=1}^n \iint\limits_{E(\frac{\rho}{2},2\rho)} u^{m_i-1}\left|u_{x_i}\right|^2\psi_r^l\, dx\, dt + \sum_{i=1}^n \iint\limits_{E(2\rho)} u^{(\rho)}\left|u_{x_i}\right|^{q_i}\psi_r^l\, dx\, dt\\
&\leq \gamma(R(r))^{nm_1}\sum_{i=1}^n M_{\frac{\rho}{2}}^{m_i+1} + \gamma F_5(r,\lambda)\left(\sum_{i=1}^n \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{\frac{1-\varepsilon_3}{q-1+\varepsilon_3}} \left(M_{\frac{\rho}{2}} - M_{2\rho}\right)^{\varepsilon_3\bar q'}
\end{aligned}$$

$$
+\gamma\left[\ln\frac{1}{R(r)}\right]^{-q\left(\frac{\varepsilon_5}{1-\varepsilon_3}-\frac{1}{n}\right)-q_1}(R(r))^{n(1-\varepsilon_3)q_1}\left(\sum_{i=1}^{n}\left(M_{\frac{\rho}{2}}-M_{2\rho}\right)^{\frac{q_i}{q_1}-1}\right)^{-1}\sum_{i=1}^{n}\left(M_{\frac{\rho}{2}}-M_{2\rho}\right)^{\frac{\varepsilon_3 q_i}{q_1}}
$$

$$
+\gamma\left(M_{\frac{\rho}{2}}-M_{2\rho}\right)\left(F_1(r,\lambda)+F_2(r,\lambda)+F_3(r,\lambda)+\rho^2\ln^2\frac{1}{\rho}\right)^{\frac{1}{2}}F_4^{\frac{1}{2}}(r,\lambda)
$$

$$
+\gamma\left(M_{\frac{\rho}{2}}-M_{2\rho}\right)H_1^{\frac{1}{2}}(r,\rho,\lambda)\rho^{\frac{2+n-\theta n+\theta\lambda}{2}}+\gamma\rho^{2-n+2\lambda}. \tag{3.27}
$$

In the equality (3.27) we will pass to the limit as $r \to 0$. By our choices

$$
\lim_{r\to 0}F_1(r,\lambda)F_4(r,\lambda)=\lim_{r\to 0}F_5(r,\lambda)=\lim_{r\to 0}F_6(r,\lambda)=0.
$$

Using (3.2) we deduce that for $\lambda = 0$ and $(\varepsilon_1-\varepsilon_2)q' \le 1$ the following relation is valid

$$
F_2(r,0)F_4(r,0)=\left[\ln\frac{1}{r}\right]^{1-(\varepsilon_1-\varepsilon_2)q'}\ln^{-1}\frac{1}{R(r)}=\left[\ln\frac{1}{r}\right]^{1-(\varepsilon_1-\varepsilon_2)q'-\delta},
$$

similarly, for $\lambda = 0$, and $(1-\varepsilon_1)q_1 \le 1$, we have

$$
F_3(r,0)F_4(r,0)=\left[\ln\frac{1}{r}\right]^{1-(1-\varepsilon_1)q_1-q\delta\left(\varepsilon_2-\frac{1}{n}\right)-\delta},
$$

choose δ from the condition

$$
\max\left(\frac{1}{2},\ 1-(\varepsilon_1-\varepsilon_2)q',\ 1-(1-\varepsilon_1)q_1,\ \frac{\varepsilon_1 q_n-1}{q\left(\varepsilon_2-\frac{1}{n}\right)}\right)<\delta<1,
$$

now passing to the limit as $r \to 0$ in (3.27) and using inequality (3.3) we complete the proof of Lemma 2.3. □

3.2 | Pointwise estimates of solutions

Set $I'=\left\{i=\overline{1,i_o}\,:\,m_i\le 1\right\}$, $I''=\left\{i=\overline{i_0+1,n}\,:\,m_i>1\right\}$, $m'=\frac{1}{n}\sum_{i=1}^{i_0}m_i$, $m''=\frac{1}{n}\sum_{i=i_0+1}^{n}m_i$, and $i_0=0$ if I' is empty, $i_0=n$ if I'' is empty. Let $(\bar x,\bar t)\in G_\lambda(R_0)\setminus G_\lambda(\rho)$ be an arbitrary point, similarly to (2.2), using the De Giorgi type iteration, we prove the following estimate

$$
\left(u(\bar x,\bar t)-M_{2\rho}\right)_+^{4+n-\left(1-m''\right)\frac{n}{2}}\le\gamma M_{\frac{\rho}{2}}^{(1-m')\frac{n}{2}}\left(\frac{M_{\frac{\rho}{2}}^2}{\rho^{\kappa(\lambda)}}+\sum_{i=1}^{n}\frac{M_{\frac{\rho}{2}}^{m_i+1}}{\rho^{2a_i(\lambda)}}\right)^{\frac{n+2}{2}}\iint\limits_{Q_{\frac{\rho}{2},\left(\frac{\rho}{2}\right)^{\kappa(\lambda)}}(\bar x,\bar t)}u_{2\rho}^2\,dx\,dt,
$$

where $Q_{\rho,\rho^{\kappa(\lambda)}}(\bar x,\bar t)=B_\rho(\bar x)\times\left(\bar t-\rho^{\kappa(\lambda)},\bar t+\rho^{\kappa(\lambda)}\right)$.

From this, since $(\bar x,\bar t)$ is an arbitrary point in $G_\lambda(R_0)\setminus G_\lambda(\rho)$, using (3.3) we obtain

$$
\left(M_\rho-M_{2\rho}\right)^{4+n-(1-m'')\frac{n}{2}}\le\gamma\rho^{-(n-\lambda)(1-m')\frac{n}{2}-(2+(n-\lambda)(m+1))\frac{n+2}{2}}\iint\limits_{Q_{\frac{\rho}{2},\left(\frac{\rho}{2}\right)^{\kappa(\lambda)}}(\bar x,\bar t)}u_{2\rho}^2\,dx\,dt.
$$

Since $u_{2\rho}\le u^{(\rho)}$ on $G_\lambda(R_0)\setminus G_\lambda\left(\frac{\rho}{2}\right)$, by Lemma 2.1 with $\alpha_i=2\frac{m_i-m_1}{m_1+1}\ge 0$, $i=\overline{1,n}$ and Lemma 3.3 we get

$$
\iint\limits_{Q_{\frac{\rho}{2},\left(\frac{\rho}{2}\right)^{\kappa(\lambda)}}(\bar x,\bar t)}u_{2\rho}^2\,dx\,dt\le\gamma\rho^{2-(n-\lambda)(1-m)}\sum_{i=1}^{n}\iint\limits_{E\left(\frac{\rho}{2},2\rho\right)}u^{m_i-1}\left|u_{x_i}\right|^2dx\,dt\le\gamma\rho^{4-n-(n-\lambda)(1-m)+2\lambda}.
$$

From the previous we obtain

$$M_\rho - M_{2\rho} \le \gamma \rho^{-n+\lambda+\lambda_0}, \quad \text{with} \quad \lambda_0 = \frac{2}{4+n-(1-m'')\frac{n}{2}} > 0.$$

Iterating the last inequality we arrive at

Theorem 3.4. *Let all the conditions of Theorem 1.1 be fulfilled. Then*

$$M_\rho(\lambda) \le \gamma \rho^{-n+\lambda+\lambda_0}. \tag{3.28}$$

3.3 | Boundedness of solutions

Let $\xi_r := \xi_r(x,t) \in C^\infty\big(R_+^{n+1}\big)$, $\xi_r = 0$ for $\rho_\lambda(x,t) \le r$, $\xi_r = 0$ for $\rho_\lambda(x,t) \ge 2r$, $0 \le \xi_r \le 1$, and $\left|\frac{\partial \xi_r}{\partial t}\right| \le \gamma r^{-\kappa(\lambda)}$, $\left|\frac{\partial \xi_r}{\partial x_i}\right| \le \gamma r^{a_i(\lambda)}$, $i = \overline{1,n}$. For $j = 0,1,2,\dots$, set $\rho_j = \frac{R_0}{2}(1+2^{-j})$, $\bar\rho_j = \frac{1}{2}\big(\rho_j + \rho_{j+1}\big)$, $k_j = k(1-2^{-j})$, $\bar k_j = \frac12\big(k_j + k_{j+1}\big)$, $A_{k_j,\rho_j} = G_\lambda\big(\rho_j\big) \cap \{u > k_j\}$, where k is positive number which will be specified later. Let $\varphi_j \in C_0^\infty\big(G_\lambda\big(\bar\rho_j\big)\big)$, $\varphi_j = 1$ in $G_\lambda\big(\rho_j\big)$, $0 \le \varphi_j \le 1$, and $\left|\frac{\partial \varphi_j}{\partial t}\right| \le \gamma 2^{j\gamma} R_0^{-\kappa(\lambda)}$, $\left|\frac{\partial \varphi_j}{\partial x_i}\right| \le \gamma 2^{j\gamma} R_0^{a_i(\lambda)}$, $i = \overline{1,n}$.

Test (1.9) by $\varphi = \big(u^\varepsilon - \bar k_j^\varepsilon\big)_+ \varphi_j^l \xi_r^{l-p}$, $\psi = \xi_r$, where $\varepsilon > 0$ is small enough to be determined later. Using (1.3) and the Young inequality we get

$$\begin{aligned}
&\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{A_{\bar k_j,\bar\rho_j}\times\{t\}} \int\limits_{\bar k_j}^{u} \big(s^\varepsilon - \bar k_j^\varepsilon\big)\, ds\, \varphi_j^l \xi_r^l\, dx + \sum_{i=1}^n \iint\limits_{A_{\bar k_j,\bar\rho_j}} u^{m_i+\varepsilon-2} \big|u_{x_i}\big|^2 \varphi_j^l \xi_r^l\, dx\, dt \\
&\quad + \sum_{i=1}^n \iint\limits_{A_{\bar k_j,\bar\rho_j}} \big(u^\varepsilon - \bar k_j^\varepsilon\big)_+ \big|u_{x_i}\big|^{q_i} \varphi_j^l \xi_r^l\, dx\, dt \\
&\le \gamma \iint\limits_{A_{\bar k_j,\bar\rho_j}} \int\limits_{\bar k_j}^{u} \big(s^\varepsilon - \bar k_j^\varepsilon\big)\, ds \left|\frac{\partial \xi_r}{\partial t}\right| \varphi_j^l \xi_r^{l-1}\, dx\, dt + \gamma \sum_{i=1}^n \iint\limits_{A_{\bar k_j,\bar\rho_j}} u^{m_i-\varepsilon}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^2 \left|\frac{\partial \xi_r}{\partial x_i}\right|^2 \varphi_j^l \xi_r^{l-2}\, dx\, dt \\
&\quad + \gamma \iint\limits_{A_{\bar k_j,\bar\rho_j}} \int\limits_{\bar k_j}^{u} \big(s^\varepsilon - \bar k_j^\varepsilon\big)\, ds \left|\frac{\partial \varphi_j}{\partial t}\right| \varphi_j^{l-1} \xi_r^l\, dx\, dt + \gamma \sum_{i=1}^n \iint\limits_{A_{\bar k_j,\bar\rho_j}} u^{m_i-\varepsilon}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^2 \left|\frac{\partial \varphi_j}{\partial x_i}\right|^2 \varphi_j^{l-2} \xi_r^l\, dx\, dt \\
&\quad + \gamma \iint\limits_{A_{\bar k_j,\bar\rho_j}} u^{m-\varepsilon}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^2 \varphi_j^l \xi_r^l\, dx\, dt.
\end{aligned} \tag{3.29}$$

Let us estimate the first two terms on the the right-hand side of (3.29). By Theorem 3.4 we obtain

$$\begin{aligned}
&\iint\limits_{A_{\bar k_j,\bar\rho_j}} \int\limits_{\bar k_j}^{u} \big(s^\varepsilon - \bar k_j^\varepsilon\big)\, ds \left|\frac{\partial \xi_r}{\partial t}\right| \varphi_j^l \xi_r^{l-1} dx\, dt + \sum_{i=1}^n \iint\limits_{A_{\bar k_j,\bar\rho_j}} u^{m_i-\varepsilon}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^2 \left|\frac{\partial \xi_r}{\partial x_i}\right|^2 \varphi_j^l \xi_r^{l-2} dx\, dt \\
&\qquad \le \gamma r^{\lambda_0(\varepsilon+1)-\varepsilon n} + \gamma r^{\lambda_0(m_1+\varepsilon)-\varepsilon n}.
\end{aligned}$$

Set $m_* = \max\big(m_n, \max_{1\le i\le n} q_i\big) + \varepsilon$, $q_* = (\varepsilon+1)\frac{q}{n} + q + \varepsilon$, if $0 < \varepsilon < \frac{\lambda_0 m^-}{n}$, then by the evident inequalities $u \le \gamma 2^{j\gamma}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^{\frac{1}{\varepsilon}}$ on $A_{\bar k_j,\bar\rho_j}$, $\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+ \le \big(u - \bar k_j\big)_+^\varepsilon$ and $\big(u - k_{j+1}\big)_+ \le \gamma\big(u^\varepsilon - k_{j+1}^\varepsilon\big)_+^{\frac1\varepsilon} + \gamma k_{j+1}^{1-\varepsilon}\big(u^\varepsilon - k_{j+1}^\varepsilon\big)_+ \le \gamma\big(u^\varepsilon - k_{j+1}^\varepsilon\big)_+^{\frac1\varepsilon} + \gamma 2^{j\gamma}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^{\frac{1-\varepsilon}{\varepsilon}}\big(u^\varepsilon - k_{j+1}^\varepsilon\big)_+ \le \gamma 2^{j\gamma}\big(u^\varepsilon - \bar k_j^\varepsilon\big)_+^{\frac1\varepsilon}$ passing to the limit $r \to 0$ in (3.29), we obtain

$$\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{A_{k_{j+1},\bar\rho_j}\times\{t\}} (u-k_{j+1})_+^{\varepsilon+1}\varphi_j^l\,dx + \sum_{i=1}^n \iint\limits_{A_{k_{j+1},\bar\rho_j}} (u-k_{j+1})_+^{\varepsilon}\left|u_{x_i}\right|^{q_i}\varphi_j^l\,dx\,dt$$
$$\le \gamma R_0^{-\gamma}2^{j\gamma}\left(\iint\limits_{A_{k_j,\rho_j}} (u-k_j)_+^{m_*}\,dx\,dt + \left|A_{\bar k_j},\bar\rho_j\right|\right).$$

From this, using Lemma 2.1 with $\alpha_i = \varepsilon > 0, i = \overline{1,n}$, we obtain

$$y_{j+1} = \iint\limits_{A_{k_{j+1},\bar\rho_{j+1}}} (u-k_{j+1})_+^{m_*}\,dx\,dt$$
$$\le \iint\limits_{A_{k_{j+1},\bar\rho_j}} (u-k_{j+1})_+^{m_*}\varphi_j^{lm_*}\,dx\,dt$$
$$\le \gamma\left(\iint\limits_{A_{k_{j+1},\bar\rho_{j+1}}} (u-k_{j+1})_+^{q_*}\varphi_j^{lq_*}\,dx\,dt\right)^{\frac{m_*}{q_*}} \left|A_{k_{j+1}},\bar\rho_j\right|^{1-\frac{m_*}{q_*}}$$
$$\le \left(\sup_{0<t<R_0^{\kappa(\lambda)}} \int\limits_{A_{k_{j+1},\bar\rho_j}\times\{t\}} (u-k_{j+1})_+^{\varepsilon+1}\varphi_j^l\,dx\right)^{\frac{m_* q}{nq_*}} \left|A_{k_{j+1}},\bar\rho_j\right|^{1-\frac{m_*}{q_*}}$$
$$\times\left(\sum_{i=1}^n \iint\limits_{A_{k_{j+1},\bar\rho_j}} (u-k_{j+1})_+^{\varepsilon}\left|u_{x_i}\right|^{q_i}\varphi_j^l\,dx\,dt + \sum_{i=1}^n \iint\limits_{A_{k_{j+1},\bar\rho_j}} (u-k_{j+1})_+^{\varepsilon+q_i}\left|\frac{\partial\varphi_j}{\partial x_i}\right|^{q_i}\varphi_j^{l-q_i}\,dx\,dt\right)^{\frac{m_*}{q_*}}$$
$$\le \gamma R_0^{-\gamma}2^{j\gamma}\left(y_j + \left|A_{\bar k_j,\bar\rho_j}\right|\right)^{\frac{m_* q}{nq_*}+\frac{m_*}{q_*}} \left|A_{k_{j+1},\bar\rho_j}\right|^{1-\frac{m_*}{q_*}}$$
$$\le \gamma R_0^{-\gamma}2^{j\gamma}k^{-m_*\left(1-\frac{m_*}{q_*}\right)}(1+k^{-m_*})^{\frac{m_* q}{nq_*}+\frac{m_*}{q_*}}\,y_j^{1+\frac{m_* q}{nq_*}}.$$

Due to Lemma 2.2 this inequality implies that $y_j \to 0$ as $j\to\infty$, if k satisfies the condition $k^{m_*+\frac{nq_*}{q}} = \gamma R_0^{-\gamma}y_0 + 1$, which implies

$$\operatorname{ess\,sup}\left\{u : (x,t)\in G_\lambda\left(\frac{R_0}{2}\right)\right\} \le \gamma R_0^{-\gamma}\left(\iint\limits_{G_\lambda(R_0)} u^{m_*}\,dx\,dt\right)^{\frac{1}{m_*+\frac{nq_*}{q}}} + \gamma,$$

if $0<\varepsilon<\frac{\lambda_0 m_1}{n}$, then the integral on the right-hand side of the previous inequality is finite, this completes the proof of the boundedness of u in the whole $G_\lambda\left(\frac{R_0}{2}\right)$.

3.4 | End of the proof of Theorem 1.1

Let $\varsigma \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;\overset{o}{W}{}^{1,2}(\Omega))$ be an arbitrary function, testing (1.9) by $\varphi = (u+\varepsilon)^{m^- -1}\varsigma\psi_r^{l-p}$, $\psi = \psi_r$, using (1.3), the Young inequality, the boundedness of u and passing to the limit $r \to 0$ and $\varepsilon \to 0$ we obtain

$$\sup_{0<t<T}\int_\Omega u^{1+m^-}\varsigma\,dx + \sum_{i=1}^n \iint_{\Omega_T} u^{m_i+m^- -2}\left|u_{x_i}\right|^2\varsigma\,dx\,dt \le \gamma \iint_{\Omega_T}\left(1+|\varsigma_t| + \sum_{i=1}^n \left|\varsigma_{x_i}\right|^2\right)dx\,dt.$$

Testing (1.9) by $\varphi = \varsigma\psi_r^{l-p}$, $\psi = \psi_r$, using (1.3), the previous inequality, the boundedness of u and passing to the limit $r \to 0$ we obtain

$$\sum_{i=1}^n \iint_{\Omega_T} \left|u_{x_i}\right|^{q_i}\varsigma\,dx\,dt \le \gamma \iint_{\Omega_T}\left(1+|\varsigma_t| + \sum_{i=1}^n \left|\varsigma_{x_i}\right|^2\right)dx\,dt.$$

From this, testing (1.9) by $\psi = \psi_r$, using the boundedness of solution, and passing to the limit $r \to 0$, we obtain the integral identity (1.9) with an arbitrary $\varphi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;\overset{o}{W}{}^{1,2}(\Omega))$ and $\psi \equiv 1$. Thus Theorem 1.1 is proved.

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