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Removability of an isolated singularity for solutions of anisotropic porous medium equation with absorption term

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Abstract. The removability of an isolated singularity for solutions to the quasilinear equation

$$u_t - \sum_{i=1}^{n} \left(u^{m_i-1} u_{x_i}\right)_{x_i} + f(u) = 0, u \geq 0,$$

is proved.

Keywords. Quasilinear parabolic equations, removable isolated singularity.

1. Introduction and the main result

We will study solutions to a quasilinear parabolic equation in the divergent form

$$u_t - divA(x, t, u, \nabla u) + a_0(u) = 0, \ (x, t) \in \Omega_T, \tag{1.1}$$

satisfying the initial condition

$$u(x, 0) = 0, \ \ x \in \Omega \setminus \{(0, 0)\} \tag{1.2}$$

in $\Omega_T = \Omega \times (0, T)$, $0 < T < \infty$, where Ω is a bounded domain in $R^n, n > 2$.

The qualitative behavior of solutions to elliptic equations was investigated by many authors starting from the seminal papers by Serrin (see [4–8]). In [1], Brezis and Veron proved that, for $q \geq \frac{n}{n-2}$, the isolated singularities of solutions to the elliptic equation

$$-\triangle u + u^q = 0,$$

are removable. The result on the removability of an isolated singularity for the parabolic equation

$$\frac{\partial u}{\partial t} - \triangle u + |u|^{q-1} u = 0, \ \ (x, t) \in \Omega_T \setminus \{(0, 0)\}$$

was obtained by Brezis and Friedman [2] in the case $q \geq \frac{n+2}{n}$. The anisotropic elliptic equation with absorption

$$-\sum_{i=1}^{n} \left(|u_{x_i}|^{p_i-2} u_{x_i}\right)_{x_i} + |u|^{q-1} u = 0$$

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was studied in [12]. It was proved that the isolated singularity for a solution of the this equation is removable, if

$$q \geq \frac{n(p-1)}{n-p}, \quad 1 \leq p_1 \leq \ldots \leq p_n \leq \frac{n-1}{n-p} p.$$

For quasilinear elliptic and parabolic equations of a special form with absorption similar questions were treated by many authors. A survey of their results and references can be found in Veron's monograph [14]. The removability of isolated singularities for more general elliptic and parabolic equations with absorption was established in [10] and [11].

We suppose that the functions $A = (a_1, ..., a_n)$ and a_0 satisfy the Carathéodory conditions, and the following structure conditions hold:

$$A(x,t,u,\xi)\xi \geq \nu_1 \sum_{i=1}^{n} |u|^{m_i-1}|\xi_i|^2,$$

$$|a_i(x,t,u,\xi)| \leq \nu_2 u^{\frac{m_i-1}{2}} \left(\sum_{j=1}^{n} |u|^{m_j-1}|\xi_j|^2 \right)^{\frac{1}{2}}, \quad i = \overline{1,n}, \tag{1.3}$$

$$a_0(u) \geq \nu_1 f(u),$$

with positive constants ν_1, ν_2, a continuous positive function $f(u)$, and

$$\min_{1\leq i\leq n} m_i > 1, \quad \max_{1\leq i\leq n} m_i \leq 1 + \frac{\kappa}{n}, \quad p < n, \tag{1.4}$$

where $\kappa = n(m-1)+2$, $d = \frac{1}{n}\sum\limits_{i=1}^{n} \frac{m_i}{2}$. Without loss of generality, we also assume that $m_n = \max\limits_{1\leq i\leq n} m_i$.

We write $V_{2,m}(\Omega_T)$ for the class of functions $\varphi \in C(0,T,L^2(\Omega))$ with $\sum\limits_{i=1}^{n} \iint\limits_{\Omega_T} |\varphi|^{m_i-1} |\varphi_{x_i}|^2 \, dxdt < \infty$.

We say that u is a weak solution to problem (1.1), (1.2) if, for an arbitrary $\psi \in C^1(\Omega_T)$ vanishing in a neighborhood of $\{(0,0)\}$, we have an inclusion $u\psi \in V_{2,m}(\Omega_T)$ and, for any interval $(t_1,t_2) \subset [0,T)$, the integral identity

$$\int\limits_{\Omega} u\varphi dx \Bigg|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \{-u\varphi_t + A(x,t,u,\nabla u)\nabla\varphi + a_0(u)\varphi\} \, dx\, dt = 0 \tag{1.5}$$

holds for $\varphi = \zeta\psi$ with an arbitrary $\zeta \in \overset{o}{V}_{2,m}(\Omega_T)$.

We say that a solution u to problem (1.1), (1.2) has a removable singularity at $\{(0,0)\}$ if u can be extended to $\{(0,0)\}$ so that the extension $\tilde{u}$ of u satisfies (1.5) with $\psi \equiv 1$ and $\tilde{u} \in V_{2,m}(\Omega_T)$.

Remark 1.1. Condition (1.4) implies the local boundedness of a weak solutions to Eq. (1.1) ([3]).

The main result of this paper is the following theorem.

Theorem 1.1. *Let conditions* (1.3) *and* (1.4) *be satisfied, and let u be a nonnegative weak solution to problem* (1.1), (1.2). *Assume also that $f(u) = u^q$ and*

$$q \geq m + \frac{2}{n}. \tag{1.6}$$

Then the singularity at the point $\{(0,0)\}$ is removable.

The rest of the paper contains the proof of Theorem 1.1.

2. Integral estimates of solutions

For $0 \le \lambda < n$ we define the numbers

$$\kappa(\lambda) = \frac{1}{2+(n-\lambda)(m-1)}, \ \kappa_i(\lambda) = \frac{2}{2+(n-\lambda)(m-m_i)}, \ i = \overline{1,n}.$$

Let

$$\rho_\lambda(x,t) = \left(t^{\frac{\kappa(\lambda)}{\kappa_1(\lambda)}} + \sum_{i=1}^{n} |x_i|^{\frac{\kappa_i(\lambda)}{\kappa_1(\lambda)}} \right)^{\kappa_1(\lambda)}.$$

We assume that $D_\lambda(r) = \{(x,t) : \rho_\lambda(x,t) < r\}$, $D_\lambda(R_0) \subset \Omega_T$. For $0 < r < R_0$ we set $M(r,\lambda) = \sup\limits_{D_\lambda(R_0)\setminus D_\lambda(r)} u(x,t)$, $E(r,\lambda) = \{(x,t) \in \Omega_T : u(x,t) > M(r,\lambda)\}$, $u_r(r,t,\lambda) = (u(x,t) - M(r,\lambda))_+$ and consider the function $\psi_r(x,t) = \eta_r(\rho_\lambda(x,t))$, where $\eta_r : \mathbb{R}^1 \to \mathbb{R}^1$ is a function taking the following values: $\eta_r(z) = 0$, if $z \le r$, $\eta_r(z) = 1$ if $z \ge R(r)$, and $\eta_r(z) = \left[(1-\varepsilon)\ln\ln\frac{1}{r}\right]^{-1}\left(\ln\ln\frac{1}{r} - \ln\ln\frac{1}{z}\right)$, if $r \le z \le R(r)$. Here, ε is a number from the interval $(0,1)$ specified in what follows, and $R(r)$ is defined by the equality

$$\ln\frac{1}{R(r)} = \ln^{\varepsilon}\frac{1}{r}. \tag{2.1}$$

Note that, by the evident equalities $\frac{1}{q-1} = (n-\lambda)\kappa(\lambda)$, $\frac{2}{q-m_i} = (n-\lambda)\kappa_i(\lambda)$, $i = \overline{1,n}$, with $\lambda \ge 0$ defined by

$$\lambda = n - \frac{2}{q-m}, \tag{2.2}$$

the Keller–Osserman estimate yields

$$M(r,\lambda) \le \gamma r^{\lambda-n}, \ r > 0. \tag{2.3}$$

This estimate follows from Theorems 4.1 and 4.2 (Appendix) in the case $p_1 = p_2 = \ldots = p_n = 2$.

Consider the functions $F_1(r,\lambda)$ and $F_2(r,\lambda)$ defined by the equalities

$$F_1(r,\lambda) = \begin{cases} R^\lambda(r), \ \lambda > 0, \\ \ln^{\frac{q-2}{q-1}}\frac{1}{r}, \ \lambda = 0, \ q > 2, \\ \ln\ln\frac{1}{r}, \ \lambda = 0, \ q = 2, \\ \ln^{-\frac{2-q}{q-1}}\frac{1}{r}, \ \lambda = 0, \ q < 2, \end{cases}$$

$$F_2(r,\lambda) = \begin{cases} R^\lambda(r), \ \lambda > 0, \\ \ln^{\frac{q-2m_1}{q-m_1}}\frac{1}{r}, \ \lambda = 0, \ q > 2m_1, \\ \ln\ln\frac{1}{r}, \ \lambda = 0, \ q = 2m_1, \\ \ln^{-\frac{2m_1-q}{1-m_1}}\frac{1}{r}, \ \lambda = 0, \ q < 2m_1. \end{cases}$$

To simplify the following calculations, we write $M(r)$, $E(r)$, and $u_r(x,t)$ instead of $M(r,\lambda)$, $E(r,\lambda)$, and $u_r(x,t,\lambda)$.

Lemma 2.1. *Let the assumptions of Theorem 1.1 be satisfied. Then the following estimate holds*

$$\sup_{0<t<T} \int\limits_{E(\frac{\rho}{2})\times\{t\}} \int\limits_{M(\frac{\rho}{2})}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E(\frac{\rho}{2})} u^{m_i-2} |u_{x_i}|^2 \psi_r^l dxdt$$

$$+ \iint\limits_{E(\frac{\rho}{2})} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \left(F_1(r,\lambda) + F_2(r,\lambda)\right) \quad (2.4)$$

for every $l \ge \frac{2q}{q-m_n}$ *and for every* $2r < \rho \le \frac{R_0}{2}$.

Proof. Testing (1.5) by $\varphi = \ln_+ \frac{u}{M(\frac{\rho}{2})} \psi_r^l$ and using (1.3) and the Young inequality, we get

$$\sup_{0<t<T} \int\limits_{E(\frac{\rho}{2})\times\{t\}} \int\limits_{M(\frac{\rho}{2})}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E(\frac{\rho}{2})} u^{m_i-2} |u_{x_i}|^2 \psi_r^l dxdt$$

$$+ \iint\limits_{E(\frac{\rho}{2})} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \iint\limits_{E(\frac{\rho}{2})} u \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial t}\right| \psi_r^{l-1} dxdt$$

$$+ \gamma \sum_{i=1}^{n} \iint\limits_{E(\frac{\rho}{2})} u^{m_i} \ln^2 \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^2 \psi_r^{l-2} dxdt.$$

From whence, by the Young inequality, we obtain

$$\sup_{0<t<T} \int\limits_{E(\frac{\rho}{2})\times\{t\}} \int\limits_{M(\frac{\rho}{2})}^{u} \ln_+ \frac{s}{M\left(\frac{\rho}{2}\right)} ds\, \psi_r^l\, dx + \sum_{i=1}^{n} \iint\limits_{E(\frac{\rho}{2})} u^{m_i-2} |u_{x_i}|^2 \psi_r^l dxdt$$

$$+ \iint\limits_{E(\frac{\rho}{2})} u^q \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \psi_r^l dxdt \le \gamma \iint\limits_{E(\frac{\rho}{2})} \ln \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial t}\right|^{\frac{q}{q-1}} dxdt$$

$$+\gamma \iint\limits_{E(\frac{\rho}{2})} \ln^{\frac{2q-m_i}{q-m_i}} \frac{u}{M\left(\frac{\rho}{2}\right)} \left|\frac{\partial \psi_r}{\partial x_i}\right|^{\frac{2q}{q-m_i}} dxdt = \gamma \left(J_1 + J_2\right). \quad (2.5)$$

By (2.3), we have

$$J_1 + J_2 \le \gamma \iint\limits_{D_\lambda(R(r))\setminus D_\lambda(r)} \ln^{-\frac{1}{q-1}} \frac{1}{\rho_\lambda}\ \rho_\lambda^{-\frac{1}{\kappa(\lambda)}\frac{q}{q-1}} dxdt$$

$$+ \gamma \sum_{i=1}^{n} \iint\limits_{D_\lambda(R(r))\setminus D_\lambda(r)} \ln^{-\frac{m_i}{q-m_i}} \frac{1}{\rho_\lambda}\ \rho_\lambda^{-\frac{2q}{\kappa_i(\lambda)(q-m_i)}} dxdt$$

$$\le \gamma \int\limits_{r}^{R(r)} \ln^{-\frac{1}{q-1}} \frac{1}{z}\ z^{\lambda-1} dz + \gamma \int\limits_{r}^{R(r)} \ln^{-\frac{m_1}{q-m_1}} \frac{1}{z}\ z^{\lambda-1} dz \le \gamma \left(F_1(r,\lambda) + F_2(r,\lambda)\right). \quad (2.6)$$

Combining (2.5) and (2.6), we obtain (2.4), which completes the proof of the lemma. □

We now define a function $u^{(\rho)}(x,t)$ and a set $E\left(\frac{\rho}{2},2\rho\right)$ as follows:

$$u^{(\rho)}(x,t)=\min\left(M\left(\frac{\rho}{2}\right)-M(2\rho),\, u_{2\rho}(x,t)\right),$$

$$E\left(\frac{\rho}{2},2\rho\right)=\{x\in E(2\rho): u<M\left(\frac{\rho}{2}\right)\}.$$

Lemma 2.2. *Under the assumptions of Lemma 2.1, the following inequality holds:*

$$\iint\limits_{E(2\rho)} u^{(\rho)}u^q\psi_r^l dxdt \le \gamma\left(M\left(\frac{\rho}{2}\right)-M(2\rho)\right)\left\{F_3(r,\lambda)+(F_1(r,\lambda)+F_2(r,\lambda))^{\frac{1}{2}}F_4^{\frac{1}{2}}(r,\lambda)\right\}, \tag{2.7}$$

where

$$F_3(r,\lambda)=\begin{cases} R^\lambda(r),\ \lambda>0,\\ \ln^{-\frac{1}{q-1}}\dfrac{1}{r},\ \lambda=0,\end{cases}\quad F_4(r,\lambda)=\begin{cases} R^\lambda(r),\ \lambda>0,\\ \ln^{-1}\dfrac{1}{r},\ \lambda=0.\end{cases}$$

Proof. Testing (1.5) by $\varphi=u^{(\rho)}\psi_r^l$ and using (1.3) and the Young inequality, we get

$$\iint\limits_{E(2\rho)} u^{(\rho)}u^q\psi_r^l dxdt \le \gamma\iint\limits_{E(2\rho)} u^{(\rho)}\left|\frac{\partial\psi_r}{\partial t}\right|^{\frac{q}{q-1}} dxdt$$

$$+\gamma\sum_{i=1}^n\iint\limits_{E(2\rho)}\left(\sum_{j=1}^n u^{m_j-1}|u_{x_j}|^2\right)^{\frac{1}{2}} u^{\frac{m_i-1}{2}}u^{(\rho)}\left|\frac{\partial\psi_r}{\partial x_i}\right|\psi_r^{l-1}dxdt$$

$$=\gamma\left(J_3+J_4\right). \tag{2.8}$$

By the Hölder inequality, (2.3), and Lemma 2.1, the integrals on the right-hand side of (2.8) are estimated as follows:

$$J_3\le\gamma\left(M\left(\frac{\rho}{2}\right)-M(2\rho)\right)\iint\limits_{E(2\rho)}\left|\frac{\partial\psi_r}{\partial t}\right|^{\frac{q}{q-1}}dxdt$$

$$\le\gamma\left(M\left(\frac{\rho}{2}\right)-M(2\rho)\right)\int\limits_{D_\lambda(R(\lambda))\setminus D_\lambda(r)}\ln^{-\frac{q}{q-1}}\frac{1}{\rho_\lambda}\ \rho_\lambda^{-\frac{q}{(q-1)\kappa(\lambda)}}dxdt$$

$$\le\gamma\left(M\left(\frac{\rho}{2}\right)-M(2\rho)\right)\int\limits_r^{R(\lambda)}\ln^{-\frac{q}{q-1}}\frac{1}{z}\,z^{\lambda-1}dz\le\gamma\left(M\left(\frac{\rho}{2}\right)-M(2\rho)\right)F_3(r,\lambda). \tag{2.9}$$

Similarly,

$$J_4 \le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \iint\limits_{E(2\rho)} u^{m_j-2} |u_{x_j}|^2 \psi_r^l dxdt \right)^{\frac{1}{2}}$$

$$\times \left(\iint\limits_{E(2\rho)} u^{m_i} \left| \frac{\partial \psi_r}{\partial x_i} \right|^2 \psi_r^l dxdt \right)^{\frac{1}{2}} \le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right)$$

$$\times (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} \sum_{i=1}^{n} \left(\iint\limits_{D_\lambda(R(\lambda))\setminus D_\lambda(r)} \ln^{-2} \frac{1}{\rho_\lambda} \rho_\lambda^{-m_i(n-\lambda) - \frac{2}{\kappa_i(\lambda)}} dxdt \right)^{\frac{1}{2}}$$

$$\le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} \left(\int\limits_{r}^{R(r)} \ln^{-2} \frac{1}{z} \; z^{\lambda-1} dz \right)^{\frac{1}{2}}$$

$$\le \gamma \left(M\left(\frac{\rho}{2}\right) - M(2\rho)\right) (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda). \tag{2.10}$$

Combining (2.8)–(2.10), we arrive at the required relation (2.7), which proves the lemma. □

2.1. Pointwise estimates of solutions

Similarly to [13], using the De Giorgi-type iteration, we prove the following estimate

$$(M(\rho) - M(2\rho)^{1+m+m\frac{n+2}{2}} \le \gamma \left(M\left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i}\left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right)^{\frac{n+2}{2}} \iint\limits_{D_\lambda(R_0)\setminus D_\lambda\left(\frac{\rho}{2}\right)} u_{2\rho}^{1+m} dxdt.$$

We note that $u_{2\rho} \le M\left(\frac{\rho}{2}\right) - M(2\rho)$ for $(x,t) \in D_\lambda(R_0) \setminus D_\lambda\left(\frac{\rho}{2}\right)$. By the Hölder inequality and Lemma 2.2, we get

$$(M(\rho) - M(2\rho)^{1+m+m\frac{n+2}{2}} \le \gamma M^{\frac{m+1}{q+1}}\left(\frac{\rho}{2}\right) \left(M\left(\frac{\rho}{2}\right) \rho^{-\frac{1}{\kappa(\lambda)}} + \sum_{i=1}^{n} M^{m_i}\left(\frac{\rho}{2}\right) \rho^{-\frac{2}{\kappa_i(\lambda)}} \right)^{\frac{n+2}{2}}$$

$$\times \left\{ F_3(r,\lambda) + (F_1(r,\lambda) + F_2(r,\lambda))^{\frac{1}{2}} F_4^{\frac{1}{2}}(r,\lambda) \right\} |D_\lambda(R_0)|^{\frac{q-m}{q+1}}. \tag{2.11}$$

In inequality (2.11) we pass to the limit as $r \to 0$. By (2.1) the following relations are valid for $\lambda = 0$:

$$F_1(r,0)F_4(r,0) = \ln^{\frac{q-2}{q-1}} \frac{1}{r} \; \ln^{-1} \frac{1}{R(r)} = \ln^{\frac{q-2}{q-1} - \varepsilon} \frac{1}{r}, \text{ if } q > 2,$$

$$F_2(r,0)F_4(r,0) = \ln^{\frac{q-2m_1}{q-m_1}} \frac{1}{r} \; \ln^{-1} \frac{1}{R(r)} = \ln^{\frac{q-2m_1}{q-m_1} - \varepsilon} \frac{1}{r}, \text{ if } q > 2m_1.$$

Let us choose ε from the condition $\max\left(\frac{1}{2}, \frac{q-2}{q-1}, \frac{q-2m_1}{q-m_1}\right) < \varepsilon < 1$. Passing to the limit as $r \to 0$ in (2.11), we obtain, for any $\rho \le \frac{R_0}{2}$,

$$M(\rho) - M(2\rho) \le 0.$$

Iterating the last inequality, we get, for any $\rho \le \frac{R_0}{2}$,

$$M(\rho) \le M(R_0).$$

This proves the boundedness of solutions.

3. End of the proof of Theorem 1.1

Let K be a compact subset in Ω, and let $\xi = 0$ in $\partial\Omega \times (0,T)$ such that $\xi = 1$ for $(x,t) \in K \times (0,T)$. Testing (1.5) by $\varphi = u\xi^2\psi_r$, $\psi = \psi_r$, using conditions (1.3), the Young inequality, and the boundedness of u and passing to the limit $r \to 0$, we get

$$\sup_{0<t<T} \int\limits_K u^2 dx + \sum_{i=1}^n \int\limits_0^T \int\limits_K u^{m_i-1}|u_{x_i}|^2 dxdt + \int\limits_0^T \int\limits_K u^{q+1} dxdt \le \gamma. \tag{3.1}$$

Testing (1.5) by $\varphi\psi_r$, where φ is an arbitrary function that belongs to $\overset{o}{V}_{2,m}(\Omega_T)$, using (3.1) and the boundedness of the solution, and passing to the limit $r \to 0$, we obtain the integral identity (1.5) with an arbitrary $\varphi \in \overset{o}{V}_{2,m}(\Omega_t)$ and $\psi \equiv 1$. Thus, Theorem 1.1 is proved.

4. Appendix

Let $(x^{(0)}, t^{(0)}) \in \Omega_T$. For any $\tau, \theta_1, \theta_2, \dots, \theta_n > 0$, $\theta = (\theta_1, \dots, \theta_n)$, we define $Q_{\theta,\tau}(x^{(0)}, t^{(0)}) := \{(x,t) : |t - t^{(0)}| < \tau,\ |x_i - x_i^{(0)}| < \theta_i,\ i = \overline{1,n}\}$ and set

$$M(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} u, \delta(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \delta(u),$$

$$\Phi(\theta,\tau) := \sup_{Q_{\theta,\tau}(x^{(0)},t^{(0)})} \Phi(u), \Phi(u) = \int\limits_0^u \varphi(s)ds, \varphi(s) = s^{m_n-1} f(s).$$

We say that a nondecreasing continuous function ψ satisfies condition (A) if for any $\varepsilon \in (0,1)$ there exists $u_0(\varepsilon) \ge 1$ such that

$$\psi(\varepsilon u) \le \varepsilon^\mu \psi(u), \tag{A}$$

with some $\mu > 0$ and for all $u \ge u_0(\varepsilon)$.

Theorem 4.1 ([9]). *Let conditions* (1.3) *and* (1.4) *be satisfied, and let* u *be a nonnegative weak solution to Eq.* (1.1). *Assume also that* $f \in C^1(R^1_+)$ *and* $f'(u) \ge 0$. *Let* $(x^{(0)}, t^{(0)}) \in \Omega_T$, *and let us fix* $\sigma \in (0,1)$, $\tau \in (0, \min(\theta_n^{p_n}, t^{(0)}, T - t^{(0)}))$, $\theta_i \in (0, \theta_n)$ *for* $i \in I' = \{i = \overline{1,n} : m_i(p_i - 1) < m_n(p_n - 1)\}$ *and* $\theta_i = \theta_n$ *for* $i \in I'' = \{i = \overline{1,n} : m_i(p_i - 1) = m_n(p_n - 1)\}$. *Then there exist positive numbers* c_8, c_9 *depending only on* $n, \nu_1, \nu_2, m_1, \dots, m_n, p_1, \dots, p_n$ *such that either*

$$u(x^{(0)}, t^{(0)}) \le (\tau^{-1}\rho^{p_n})^{\frac{1}{m_n(p_n-1)-1}} + \sum_{i \in I'} (\theta_i^{-1}\theta_n^{\frac{p_n}{p_i}})^{\frac{p_i}{m_n(p_n-1)-m_i(p_i-1)}} \tag{4.1}$$

or

$$\Phi(\sigma\theta, \sigma\tau) \le c_8(1-\sigma)^{-c_9}\theta_n^{-p_n}\delta(\theta,\tau)M^{m_np_n-1}(\theta,\tau). \tag{4.2}$$

On the other hand, if I' is empty, i.e. $m_1(p_1-1)=m_2(p_2-1)=\cdots=m_n(p_n-1)$, then either

$$u(x^{(0)},t^{(0)})\le(\tau^{-1}\theta_n^{p_n})^{\frac{1}{m_n(p_n-1)-1}} \tag{4.3}$$

or (4.2) *holds true.*

Theorem 4.2 ([9]). *Let conditions* (1.3) *and* (1.4) *be satisfied, and let u be a nonnegative weak solution to* (1.1)*, $f\in C^1(R^1_+)$, and $f'(u)\ge 0$. Let $\partial\Omega_T$ be the parabolic boundary of Ω_T. Assume also that $\lim\limits_{(x,t)\to\partial\Omega_T} u(x,t)=+\infty$, and, with some $0\le a\le 1$ and $c>0$, the following relation holds:*

$$\delta(u)\le cu^a.$$

Let $\psi(u)=u^{-1}\Phi^{\frac{1}{m_np_n+a-1}}(u)$ satisfy condition (A). Let $(x^{(0)},t^{(0)})\in\Omega_T$ and $8\rho=dist(x^{(0)},\partial\Omega)$. Fix $\tau\in(0,\min(\rho^{p_n},t^{(0)},T-t^{(0)}))$ and $\theta_i\in(0,\rho)$ for $i\in I'$. Then there exists a positive number c_{10} that depends only on $n,\nu_1,\nu_2,m_1,...,m_n,p_1,\ ...,p_n$, and c and is such that either (4.1) holds, or

$$\Phi(u(x^{(0)},t^{(0)}))\le c_{10}\theta_n^{-p_n}u^{m_np_n+a-1}(x^{(0)},t^{(0)}). \tag{4.4}$$

On the other hand, if I' is empty, i.e., $m_1(p_1-1)=m_2(p_2-1)=...=m_n(p_n-1)$, and $\psi(u)=u^{-1}\Phi^{\frac{1}{m_np_n+a-1}}(u)$ satisfies condition (A), then either (4.3) *holds, or* (4.4) *holds true.*

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