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ON UNIQUENESS OF ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC DEGENERATE ANISOTROPIC EQUATIONS

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In the present paper we deal with the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides in a bounded domain of $\mathbb{R}^n$ $(n \geqslant 2)$. This class is described by the presence of a set of exponents $q_1, \ldots, q_n$ and a set of weighted functions $\nu_1, \ldots, \nu_n$ in growth and coercitivity conditions on coefficients of the equations. The exponents q_i characterize the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function, and the functions ν_i characterize degeneration or singularity of the coefficients with respect to independent variables. Our aim is to study the uniqueness of entropy solution of the problem under consideration.

Introduction. The studying of nonlinear elliptic second-order equations with L^1-data and measures as data is one of the important directions of a modern differential equation theory. In this theory the concepts of a weak solution, entropy solution, renormalized solution were introduced, the theorems on existence and uniqueness of these solutions were proved, and their belonging to Lebesgue and Sobolev spaces were established.

A weak solution (solution from $W^{1,1}$ in sense of the integral identity for smooth functions) to equations with L^1-right-hand sides is a natural analogue of a generalized solution to equations with "well enough", right-hand sides. The theorems on the existence of a weak solution to the Dirichlet problem for nonlinear elliptic equations were obtained in [5], [6]. Remark that a weak solution exists not for all values of a parameter characterizing the growth of equation's coefficients with respect to the corresponding derivatives of unknown function. In general, a weak solution is not a unique one.

An effective approach to the solvability of the Dirichlet problem for nonlinear elliptic second-order equations with L^1-right-hand sides has been proposed in [4]. There a concept of an entropy solution to the problem under consideration was introduced. This solution belongs to a new special function's class that includes the corresponding Sobolev space. Under standard growth, coercitivity and strict monotonicity conditions on the equation's coefficients authors proved the theorem on existence and uniqueness of an entropy solution to the given problem. Notice that an entropy solution is unique for all values of a parameter characterizing the growth of equation's coefficients with respect to the corresponding derivatives of unknown function.

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Above-mentioned works and other close investigations are devoted to L^1-theory for nonlinear equations with isotropic and nondegenerate (with respect to the independent variables) coefficients. As for the solvability of nonlinear elliptic second-order equations with anisotropy and degeneracy (with respect to the independent variables), we note the following works. The existence of a weak solution to the Dirichlet problem for a model nondegenerate anisotropic equation with right-hand side measure was established in [7]. The existence of weak solutions for a class of nondegenerate anisotropic equations with locally integrable data was proved in [3]. Solvability of the Dirichlet problem for degenerate isotropic equations with L^1-data and measures as data was studied in [1], [2], [8], [9], [16]. Remark that in [1], [8], the existence of entropy solutions to the given problem was proved in the case of L^1-data, and in [2], the existence of a renormalized solution of the problem for the same case was established. In [2], [9], [16], the existence of distributional solutions of the problem was obtained in the case of right-hand side measures.

Solvability of the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides was studied in [14]. This class is described by the presence of a set of exponents $q_1, \dots, q_n$ and of a set of weighted functions $\nu_1, \dots, \nu_n$ in growth and coercitivity conditions on coefficients of the equations under consideration. The exponents q_i characterize the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function, and the functions ν_i characterize degeneration or singularity of the coefficients with respect to the independent variables.

In [14], the theorem on the existence and uniqueness of entropy solution to the Dirichlet for this class of the equations was proved (see [14], Theorem 3.2). Observe that the proof of this theorem is based on use of some results of [11]–[13] on the existence and properties of solutions of second-order variational inequalities with L^1-right-hand sides and sufficiently general constraints. Note that in [11]–[14] right-hand sides to the investigated variational inequalities and equations depend on independent variables only, and belong to the class L^1.

The present paper is devoted to the Dirichlet problem for the same class of the nonlinear elliptic second-order equations in divergence form with degenerate anisotropic coefficients as in [14]. Now right-hand sides to the given equations depend on independent variables and unknown function. In our case we have no opportunity to use the results [11]–[13] directly. We follow a general approach for proving the main result of this work (theorem 1). As we mentioned above, this approach has been proposed in [4] to the investigation on the existence and properties of solutions for nonlinear elliptic second-order equations with isotropic nondegenerate (with respect to the independent variables) coefficients and L^1-right-hand sides. In [11], [13] this approach has been taken to the anisotropic degenerate case. Also we use some ideas of [15].

1. Preliminaries. In this section we give some results of [13] which will be used in the sequel.

Let $n \in \mathbb{N}$, $n \geqslant 2$, Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$, and let for every $i \in \{1, \dots, n\}$ we have $q_i \in (1, n)$.

We set $q = \{q_i : i = 1, \dots, n\}$,

$$\overline{q} = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{q_i} \right)^{-1}, \quad \hat{q} = \frac{n(\overline{q} - 1)}{(n-1)\overline{q}}\,.$$

Let for every $i \in \{1, \dots, n\}$ ν_i be a nonnegative function on Ω such that $\nu_i > 0$ a.e.

in Ω,

$$\nu_i \in L^1_{\rm loc}(\Omega), \quad (1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega). \tag{1}$$

We set $\nu = \{\nu_i : i = 1, \dots, n\}$. We denote by $W^{1,q}(\nu,\Omega)$ the set of all functions $u \in W^{1,1}(\Omega)$ such that for every $i \in \{1,\dots,n\}$ we have $\nu_i |\, D_i u\,|^{q_i} \in L^1(\Omega)$.

Let $\|\cdot\|_{1,q,\nu}$ be the mapping from $W^{1,q}(\nu,\Omega)$ into $\mathbb{R}$ such that for every function $u \in W^{1,q}(\nu,\Omega)$

$$\|u\|_{1,q,\nu} = \int_\Omega |u| dx + \sum_{i=1}^n \left(\int_\Omega \nu_i |D_i u|^{q_i} dx \right)^{1/q_i}.$$

The mapping $\|\cdot\|_{1,q,\nu}$ is a norm in $W^{1,q}(\nu,\Omega)$, and, in view of the second inclusion of (1), the set $W^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,q,\nu}$. Moreover, by virtue of the first inclusion of (1), we have $C_0^\infty(\Omega) \subset W^{1,q}(\nu,\Omega)$.

We denote by $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in space $W^{1,q}(\nu,\Omega)$. Evidently, the set $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm induced by the norm $\|\cdot\|_{1,q,\nu}$. It is obvious that $\overset{\circ}{W}{}^{1,q}(\nu,\Omega) \subset \overset{\circ}{W}{}^{1,1}(\Omega)$.

Further, let for every $k > 0$ $\quad T_k : \mathbb{R} \to \mathbb{R}$ be the function such that

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leqslant k, \\ k \operatorname{sign} s, & \text{if } |s| > k. \end{cases}$$

By analogy with known results for nonweighted Sobolev spaces (see for instance [10]) we have: if $u \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$, and $k > 0$, then $T_k(u) \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$, and for every $i \in \{1,\dots,n\}$,

$$D_i T_k(u) = D_i u \cdot 1_{\{|u|<k\}} \quad \text{a.e. in } \ \Omega. \tag{2}$$

We denote by $\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$ $T_k(u) \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$. Clearly, $\overset{\circ}{W}{}^{1,q}(\nu,\Omega) \subset \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$.

For every $u : \Omega \to \mathbb{R}$ and for every $x \in \Omega$ we set $k(u,x) = \min\{l \in \mathbb{N} : |u(x)| \leqslant l\}$.

Definition 1. Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, and $i \in \{1,\dots,n\}$. Then $\delta_i u : \Omega \to \mathbb{R}$ is the function such that for every $x \in \Omega$ $\quad \delta_i u(x) = D_i T_{k(u,x)}(u)\,(x)$.

Definition 2. If $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, then $\delta u : \Omega \to \mathbb{R}^n$ is the mapping such that for every $x \in \Omega$ and for every $i \in \{1,\dots,n\}$ $\quad (\delta u(x))_i = \delta_i u(x)$.

Now we give several propositions which will be used in the next sections.

Proposition 1. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$. Then for every $k > 0$ we have $D_i T_k(u) = \delta_i u \cdot 1_{\{|u|<k\}}$ a.e. in Ω, $i = 1,\dots,n$.*

Note that for every function $u \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ $\quad \delta_i u = D_i u$ a.e. in Ω, $\quad i = 1,\dots,n$.

Proposition 2. *Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, $w \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$. Then $u - w \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, and for every $i \in \{1,\dots,n\}$ and for every $k > 0$ we have*

$$D_i T_k(u-w) = \delta_i u - D_i w \quad \text{a.e. in } \ \{|u-w| < k\}.$$

Proposition 3. *There exists a positive constant c_0 depending only on n, q, $\|1/\nu_i\|_{L^{1/(q_i-1)}(\Omega)}$, $i = 1, \ldots, n$, such that for every function $u \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$*

$$\left(\int_\Omega |u|^{n/(n-1)} dx \right)^{(n-1)/n} \leqslant c_0 \prod_{i=1}^n \left(\int_\Omega \nu_i |D_i u|^{q_i} dx \right)^{1/nq_i}.$$

2. Statement of the Dirichlet problem. The concept of its entropy solution. Let c_1, $c_2 > 0$, g_1, $g_2 \in L^1(\Omega)$, g_1, $g_2 \geqslant 0$ in Ω, and let for every $i \in \{1, \ldots, n\}$ $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^n (1/\nu_i)^{1/(q_i-1)}(x) |a_i(x, \xi)|^{q_i/(q_i-1)} \leqslant c_1 \sum_{i=1}^n \nu_i(x) |\xi_i|^{q_i} + g_1(x), \tag{3}$$

$$\sum_{i=1}^n a_i(x, \xi)\xi_i \geqslant c_2 \sum_{i=1}^n \nu_i(x) |\xi_i|^{q_i} - g_2(x). \tag{4}$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^n [a_i(x, \xi) - a_i(x, \xi')] (\xi_i - \xi'_i) > 0. \tag{5}$$

Now we give one result of [14] which will be used in the sequel.

Proposition 4. *The following assertions hold:*

a) if $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$, $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, $k > 0$, $l \geqslant k + \|w\|_{L^\infty(\Omega)}$, and $i \in \{1, \ldots, n\}$, then $a_i(x, \delta u) D_i T_k(u - w) = a_i(x, \nabla T_l(u)) D_i T_k(u - w)$ a.e. in Ω;

b) if $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$, $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, $k > 0$, and $i \in \{1, \ldots, n\}$, then $a_i(x, \delta u) D_i T_k(u - w) \in L^1(\Omega)$.

Let $F \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. We consider the following Dirichlet problem:

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, \nabla u) = F(x, u) \quad \text{in } \Omega, \tag{6}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Definition 3. An *entropy solution of problem* (6), (7) is a function $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu, \Omega)$ such that:

$$F(x, u) \in L^1(\Omega); \tag{8}$$

for every function $w \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$ and for every $k \geqslant 1$,

$$\int_\Omega \left\{ \sum_{i=1}^n a_i(x, \delta u) D_i T_k(u - w) \right\} dx \leqslant \int_\Omega F(x, u)\, T_k(u - w) dx. \tag{9}$$

Note that the left-hand integral in (9) is finite. It follows from assertion b) of Proposition 4. The right-hand integral in (9) is also finite. It follows from the boundedness of the function T_k and inclusion (8).

3. On uniqueness of an entropy solution. Firstly, we prove two auxiliary results.

Lemma 1. *Let u be an entropy solution of the Dirichlet problem* (6), (7). *Then there exists a nonnegative constant M such that for every $k \geqslant 1$,*

$$\operatorname{meas}\{|u| \geqslant k\} \leqslant M k^{-\hat{q}}. \tag{10}$$

Proof. We fix an arbitrary function $v \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega) \cap L^\infty(\Omega)$, and set

$$M_* = \frac{2}{c_2}\left\{\frac{c_2}{2c_1}\|g_1\|_{L^1(\Omega)} + \|g_2\|_{L^1(\Omega)} + (2n)^{n-1}\Big(\frac{c_1}{c_2}+1\Big)^{n-1}\int_\Omega\Big\{\sum_{i=1}^n \nu_i|D_iv|^{q_i}\Big\}dx + \right.$$

$$\left. +(1+\|v\|_{L^\infty(\Omega)})\int\limits_\Omega |F(x,u)|\,dx\right\}, \quad M = \big(c_0 M_*^{1/\bar{q}}\big)^{n/(n-1)}.$$

Let $k \geqslant 1$. We put $k_1 = k + \|v\|_{L^\infty(\Omega)}$,

$$I = \sum_{i=1}^n \int_{\{|u-v|<k_1\}} \nu_i|\delta_i u|^{q_i} dx, \quad J = \sum_{i=1}^n \int_{\{|u-v|<k_1\}} |a_i(x,\delta u)|\,|D_i v|\,dx.$$

In view of (9),

$$\int_\Omega \Big\{\sum_{i=1}^n a_i(x,\delta u) D_i T_{k_1}(u-v)\Big\}dx \leqslant \int_\Omega F(x,u)\,T_{k_1}(u-v)dx.$$

Using Propositions 1 and 2, and (4), we obtain from this inequality:

$$c_2 I \leqslant k_1\int_\Omega |F(x,u)|\,dx + \|g_2\|_{L^1(\Omega)} + J.$$

On the other hand, taking into account Young's inequality and (3), we find that

$$J \leqslant \frac{c_2}{2}\,I + \frac{c_2}{2c_1}\,\|g_1\|_{L^1(\Omega)} + (2n)^{n-1}\Big(\frac{c_1}{c_2}+1\Big)^{n-1}\int_\Omega\Big\{\sum_{i=1}^n \nu_i|D_iv|^{q_i}\Big\}dx.$$

From latter two estimates it follows that

$$I \leqslant M_* k. \tag{11}$$

Further, we have $|T_k(u)| = k$ on $\{|u| \geqslant k\}$. Then

$$k^{n/(n-1)}\operatorname{meas}\{|u| \geqslant k\} \leqslant \int_\Omega |T_k(u)|^{n/(n-1)}\,dx. \tag{12}$$

Since $T_k(u) \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$, from (11), Propositions 3 and 1 we get

$$\left(\int_\Omega |T_k(u)|^{n/(n-1)}dx\right)^{(n-1)/n} \leqslant c_0\prod_{i=1}^n\left(\int_\Omega \nu_i\,|D_iT_k(u)|^{q_i}dx\right)^{1/nq_i} =$$

$$= c_0\prod_{i=1}^n\left(\int_{\{|u|<k\}} \nu_i\,|\delta_i u|^{q_i}dx\right)^{1/nq_i} \leqslant c_0\prod_{i=1}^n\left(\int_{\{|u-v|<k_1\}} \nu_i\,|\delta_i u|^{q_i}dx\right)^{1/nq_i} \leqslant$$

$$\leqslant c_0 I^{1/\bar{q}} \leqslant c_0(M_* k)^{1/\bar{q}}.$$

This estimate and (12) imply (10). □

Lemma 2. *Let u be an entropy solution of the Dirichlet problem* (6), (7). *Then for every* $v \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$, $k \geqslant 1$, $h \geqslant 1$,

$$\int_{\{h\leqslant|u-v|<h+k\}} \left\{ \sum_{i=1}^n \nu_i |\delta_i u|^{q_i} \right\} dx \leqslant \frac{2k}{c_2} \int_{\{|u-v|\geqslant h\}} |F(x,u)|\, dx + \\ + \frac{2(2n)^{n-1}}{c_2} \left(\frac{c_1}{c_2}+1\right)^{n-1} \int_{\{h\leqslant|u-v|<h+k\}} \left\{ \sum_{i=1}^n \nu_i |D_i v|^{q_i} + g_1 + g_2 \right\} dx. \tag{13}$$

Proof. We fix arbitrary $v \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega) \cap L^\infty(\Omega)$, $k \geqslant 1$, and $h \geqslant 1$.

Put

$$w = v + T_h(u-v), \qquad k_1 = k + \|w\|_{L^\infty(\Omega)}.$$

From (9) and assertion a) of Proposition 4 it follows that

$$\int_{\{|u-w|<k\}} \left\{ \sum_{i=1}^n a_i(x, \nabla T_{k_1}(u))\, D_i T_k(u-w) \right\} dx \leqslant k \int_{\{|u-v|\geqslant h\}} |F(x,u)|\, dx. \tag{14}$$

We set $G_1 = \{h \leqslant |u-v| < h+k\}$, $G_2 = \{|u-v| < h\}$. Observe that

$$\{|u-w|<k\} = G_1 \cup G_2, \qquad G_1 \cap G_2 = \varnothing. \tag{15}$$

We have

$$T_k(u-w) = T_{k_1}(u) - v - T_h(u-v) \quad \text{a.e. in } G_1, \qquad T_k(u-w) = 0 \quad \text{in } G_2.$$

Hence, for every $i \in \{1,\dots,n\}$,

$$D_i T_k(u-w) = D_i T_{k_1}(u) - D_i v \quad \text{a.e. in } G_1, \qquad D_i T_k(u-w) = 0 \quad \text{a.e. in } G_2.$$

These facts, and (14), (15) imply

$$\int_{G_1} \left\{ \sum_{i=1}^n a_i(x, \nabla T_{k_1}(u))\, D_i T_{k_1}(u) \right\} dx \leqslant \\ \leqslant \int_{G_1} \left\{ \sum_{i=1}^n a_i(x, \nabla T_{k_1}(u))\, D_i v \right\} dx + k \int_{\{|u-v|\geqslant h\}} |F(x,u)|\, dx. \tag{16}$$

We denote by I_1 the integral from the left-hand side of (16), and by I_2 the integral from the right-hand side of (16). By virtue of (4), we get

$$I_1 \geqslant c_2 \int_{G_1} \left\{ \sum_{i=1}^n \nu_i |D_i T_{k_1}(u)|^{q_i} \right\} dx - \int_{G_1} g_2\, dx.$$

From this estimate and (16) it follows that

$$c_2 \int_{G_1} \left\{ \sum_{i=1}^n \nu_i |D_i T_{k_1}(u)|^{q_i} \right\} dx \leqslant k \int_{\{|u-v|\geqslant h\}} |F(x,u)|\, dx + \int_{G_1} g_2\, dx + I_2. \tag{17}$$

Using (3) and Young's inequality, we obtain

$$I_2 \leqslant \frac{c_2}{2}\int_{G_1}\left\{\sum_{i=1}^{n}\nu_i|D_i\,T_{k_1}(u)|^{q_i}\right\}dx+\frac{c_2}{2c_1}\int_{G_1}g_1\,dx+$$
$$+(2n)^{n-1}\Big(\frac{c_1}{c_2}+1\Big)^{n-1}\int_{G_1}\left\{\sum_{i=1}^{n}\nu_i|D_iv|^{q_i}\right\}dx. \tag{18}$$

Note that in view of proposition 1 for every $i\in\{1,\dots,n\}$ we have $D_i\,T_{k_1}(u)=\delta_i u$ a.e. in G_1. Taking into account this fact, we deduce the inequality (13) from (17) and (18). □

Next theorem is the main result of this paper.

Theorem 1. *Let for a.e.* $x\in\Omega\quad F(x,\cdot)$ *be the nonincreasing function on* $\mathbb{R}$*, and let* u_1, u_2 *be entropy solutions of the Dirichlet problem* (6), (7). *Then* $u_1=u_2$ *a.e. in* Ω.

Proof. We denote by $c_i,\ i=3,4,\dots,$ the positive constants depending only on n, c_1 and c_2.

Fix an arbitrary function $v\in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)\cap L^\infty(\Omega)$, and set

$$\Phi_j=\sum_{i=1}^{n}\nu_i|D_iv|^{q_i}+g_1+g_2+|F(x,u_j)|,\quad j=1,2.$$

From Lemma 2 it follows that for every $k\geqslant 1$, $h\geqslant k+1$,

$$\int_{\{h-k\leqslant|u_j-v|<h+k\}}\left\{\sum_{i=1}^{n}\nu_i|\delta_iu_j|^{q_i}\right\}dx\leqslant c_3\,k\int_{\{|u_j-v|\geqslant h-k\}}\Phi_j\,dx,\quad j=1,2. \tag{19}$$

Fix an arbitrary $k\geqslant 1$, $h\geqslant k+1$, and put

$$G\ =\{|u_1-u_2|<k,\ |u_1-v|<h,\ |u_2-v|<h\},\quad G_1=\{|u_1-v|<h,\ |u_2-v|<h\},$$
$$G_2=\{|u_1-v|\geqslant h\}\cup\{|u_2-v|\geqslant h\},\quad w\ =v+T_h(u_2-v),\quad l=k+\|w\|_{L^\infty(\Omega)}.$$

By virtue of Definition 3 and assertion a) of Proposition 4, we have

$$\int_\Omega\left\{\sum_{i=1}^{n}a_i(x,\delta u_1)D_iT_k(u_1-w)\right\}dx=\int_\Omega\left\{\sum_{i=1}^{n}a_i(x,\nabla T_l(u_1))D_iT_k(u_1-w)\right\}dx\leqslant$$
$$\leqslant\int_{G_1}F(x,u_1)\,T_k(u_1-u_2)\,dx+k\int_{G_2}|F(x,u_1)|\,dx. \tag{20}$$

Now we estimate lower bound the left-hand side of this inequality. Put

$$E'=\{|u_1-w|<k,\ |u_2-v|<h\},\quad E''=\{|u_1-w|<k,\ |u_2-v|\geqslant h\}.$$

It is clear that

$$G\subset E'. \tag{21}$$

Besides, we have

$$E'\setminus G\subset\{h\leqslant|u_1-v|<h+k\}\cap\{h-k\leqslant|u_2-v|<h\}, \tag{22}$$

$$E'' \subset \{h-k \leqslant |u_1 - v| < h+k\}. \tag{23}$$

In fact, let $x \in E' \setminus G$. Then $|u_1(x) - u_2(x)| < k,\ |u_2(x) - v(x)| < h,\ |u_1(x) - v(x)| \geqslant h$. Hence,

$$h \leqslant |u_1(x) - v(x)| \leqslant |u_1(x) - u_2(x)| + |u_2(x) - v(x)| < k + |u_2(x) - v(x)| < h + k.$$

Inclusion (22) follows from these estimates. Let now $x \in E''$. Therefore,

$$|u_1(x) - w(x)| < k, \quad |u_2(x) - v(x)| \geqslant h. \tag{24}$$

By virtue of the second inequality of (24), and definition of the function w we have $w(x) = v(x) + h \operatorname{sign}(u_2(x) - v(x))$. So, in view of the first inequality of (24), we get

$$|u_1(x) - v(x)| \leqslant |u_1(x) - w(x)| + |w(x) - v(x)| < h + k,$$
$$h = |v(x) - w(x)| \leqslant |u_1(x) - v(x)| + |u_1(x) - w(x)| < |u_1(x) - v(x)| + k.$$

Hence, inclusion (23) is true.

Further, since

$$T_k(u_1 - w) = T_l(u_1) - T_l(u_2) \quad \text{a.e. in } \ E',$$

for every $i \in \{1, \ldots, n\}$ we have

$$D_i\, T_k(u_1 - w) = D_i\, T_l(u_1) - D_i\, T_l(u_2) \quad \text{a.e. in } \ E'. \tag{25}$$

By analogy,

$$T_k(u_1 - w) = T_l(u_1) - v - T_h(u_2 - v) \quad \text{a.e. in } \ E'',$$

thus, for every $i \in \{1, \ldots, n\}$ we have

$$D_i\, T_k(u_1 - w) = D_i\, T_l(u_1) - D_i v \quad \text{a.e. in } \ E''. \tag{26}$$

Taking into account (25) and (26), we obtain

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \nabla T_l(u_1)) D_i T_k(u_1 - w) \Big\} dx =$$
$$= \int_{E'} \Big\{ \sum_{i=1}^n a_i(x, \nabla\, T_l(u_1))\, [D_i\, T_l(u_1) - D_i T_l(u_2)] \Big\}\, dx +$$
$$+ \int_{E''} \Big\{ \sum_{i=1}^n a_i(x, \nabla\, T_l(u_1))\, [D_i\, T_l(u_1) - D_i v] \Big\}\, dx.$$

From this fact, (21), and (4) it follows that

$$\int_\Omega \Big\{ \sum_{i=1}^n a_i(x, \nabla T_l(u_1)) D_i T_k(u_1 - w) \Big\} dx \geqslant$$
$$\geqslant \int_G \Big\{ \sum_{i=1}^n a_i(x, \nabla\, T_l(u_1))\, [D_i\, T_l(u_1) - D_i T_l(u_2)] \Big\}\, dx - \int_{(E' \setminus G) \cup E''} g_2\, dx -$$
$$- \int_{E' \setminus G} \Big\{ \sum_{i=1}^n |a_i(x, \nabla\, T_l(u_1))|\, |D_i\, T_l(u_2)| \Big\}\, dx - \int_{E''} \Big\{ \sum_{i=1}^n |a_i(x, \nabla\, T_l(u_1))|\, |D_i\, v| \Big\}\, dx. \tag{27}$$

We denote by I' and I'' the third and fourth integral in the right-hand side of the latter estimate correspondingly. Using Young's inequality and (3), we get

$$I' \leqslant c_1 \int_{E'\setminus G} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, T_l(u_1)|^{q_i} \right\} dx + \int_{E'\setminus G} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, T_l(u_2)|^{q_i} \right\} dx + \int_{E'\setminus G} g_1 \, dx, \quad (28)$$

$$I'' \leqslant c_1 \int_{E''} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, T_l(u_1)|^{q_i} \right\} dx + \int_{E''} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, v|^{q_i} \right\} dx + \int_{E''} g_1 \, dx. \quad (29)$$

In view of Proposition 1, inclusions (22) and (23), and inequality (19) we have

$$\int_{(E'\setminus G)\cup E''} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, T_l(u_1)|^{q_i} \right\} dx =$$

$$= \int_{(E'\setminus G)\cup E''} \left\{ \sum_{i=1}^{n} \nu_i \, |\delta_i u_1|^{q_i} \right\} dx \leqslant c_3 \, k \int_{\{|u_1 - v| \geqslant h-k\}} \Phi \, dx, \quad (30)$$

$$\int_{E'\setminus G} \left\{ \sum_{i=1}^{n} \nu_i \, |D_i \, T_l(u_2)|^{q_i} \right\} dx = \int_{E'\setminus G} \left\{ \sum_{i=1}^{n} \nu_i \, |\delta_i u_2|^{q_i} \right\} dx \leqslant c_3 \, k \int_{\{|u_2 - v| \geqslant h-k\}} \Phi \, dx. \quad (31)$$

From (28)–(31) and (22), (23) we infer that

$$\int_{(E'\setminus G)\cup E''} g_2 \, dx + I' + I'' \leqslant c_4 \, k \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}.$$

Using this inequality and (27), we deduce that

$$\int_{\Omega} \left\{ \sum_{i=1}^{n} a_i(x, \nabla T_l(u_1)) D_i T_k(u_1 - w) \right\} dx \geqslant$$

$$\geqslant \int_{G} \left\{ \sum_{i=1}^{n} a_i(x, \nabla \, T_l(u_1)) \, [D_i \, T_l(u_1) - D_i \, T_l(u_2)] \right\} dx -$$

$$- c_4 \, k \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}. \quad (32)$$

In view of Proposition 1 $\nabla \, T_l(u_j) = \delta u_j$ a.e. in G, $j = 1, 2$.
This fact, (32) and (20) imply

$$\int_{G} \left\{ \sum_{i=1}^{n} a_i(x, \delta u_1) \, [\delta_i u_1 - \delta_i u_2] \right\} dx \leqslant$$

$$\leqslant \int_{G_1} F(x, u_1) \, T_k(u_1 - u_2) \, dx + c_5 \, k \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}.$$

By analogy we have

$$\int_{G} \left\{ \sum_{i=1}^{n} a_i(x, \delta u_2) \, [\delta_i u_2 - \delta_i u_1] \right\} dx \leqslant$$

$$\leqslant \int_{G_1} F(x, u_2) \, T_k(u_2 - u_1) \, dx + c_5 \, k \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}.$$

Adding two latter inequalities, we establish that for every $k \geqslant 1$, $h \geqslant k+1$,

$$\int_{\{|u_1-u_2|<k,|u_1-v|<h,|u_2-v|<h\}} \left\{ \sum_{i=1}^{n} [a_i(x,\delta u_1) - a_i(x,\delta u_2)]\,[\delta_i u_1 - \delta_i u_2] \right\} dx \leqslant$$
$$\leqslant \int_{\{|u_1-v|<h,|u_2-v|<h\}} [F(x,u_1) - F(x,u_2)]T_k(u_1-u_2)\,dx +$$
$$+2\,c_5\,k \left\{ \int_{\{|u_1-v|\geqslant h-k\}} \Phi_1\,dx + \int_{\{|u_2-v|\geqslant h-k\}} \Phi_2\,dx \right\}. \tag{33}$$

As for a. e. $x \in \Omega$ the function $F(x,\cdot)$ is nonincreasing on $\mathbb{R}$, we have

$$[F(x,u_1) - F(x,u_2)]T_k(u_1-u_2) \leqslant 0 \quad \text{a.e. in } \Omega. \tag{34}$$

From Lemma 1 it follows that

$$\operatorname{meas}\{|u_j - v| \geqslant h-k\} \to 0,\ h \to +\infty,\ k \geqslant 1,\ j=1,2.$$

This fact imply

$$\forall k \geqslant 1 \quad \int_{\{|u_j-v|\geqslant h-k\}} \Phi_j\,dx \to 0, \quad h \to +\infty,\ j=1,2. \tag{35}$$

Taking into account (34), (5), and using Fatou's lemma, we infer from (33), (35)

$$\delta u_1 = \delta u_2 \quad \text{a. e. in } \ \Omega. \tag{36}$$

Let again $k \geqslant 1$, $h \geqslant k+1$. Put

$$z = T_h(u_1 - v) - T_h(u_2 - v).$$

Clearly, $z \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$. Hence, $T_k(z) \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$. In view of Proposition 3 and Young's inequality we have

$$\left(\int_\Omega |T_k(z)|^{n/(n-1)} dx \right)^{(n-1)/n} \leqslant c_0 \sum_{i=1}^{n} \left(\int_\Omega \nu_i\,|D_i\,T_k(z)|^{q_i} dx \right)^{1/q_i}. \tag{37}$$

Let

$$H_1 = \{|z|<k, |u_1-v|<h, |u_2-v|<h\}, \quad H_2 = \{|z|<k, |u_1-v|<h, |u_2-v|\geqslant h\},$$
$$H_3 = \{|z|<k, |u_1-v|\geqslant h, |u_2-v|<h\}, \quad H_4 = \{|z|<k, |u_1-v|\geqslant h, |u_2-v|\geqslant h\}.$$

It is obvious that

$$H_m \cap H_r = \varnothing, \quad m \neq r, \quad m,r = 1,\dots,4, \quad \{|z|<k\} = \bigcup_{m=1}^{4} H_m. \tag{38}$$

We fix an arbitrary $i \in \{1,\dots,n\}$. Taking into account (2) and (38), we obtain

$$\int_\Omega \nu_i\,|D_i\,T_k(z)|^{q_i}\,dx = \sum_{m=1}^{4} \int_{H_m} \nu_i\,|D_i z|^{q_i}\,dx. \tag{39}$$

From Proposition 2 and (36) we get

$$D_i z = 0 \quad \text{a.e. in} \quad H_1. \tag{40}$$

It is easy to show that

$$H_2 \subset \{h - k < |u_1 - v| < h\}, \quad H_3 \subset \{h - k < |u_2 - v| < h\}. \tag{41}$$

Besides, in view of Propositions 1 and 2,

$$D_i z = \delta_i u_1 - D_i v \quad \text{a.e. in} \quad H_2, \tag{42}$$

$$D_i z = D_i v - \delta_i u_2 \quad \text{a.e. in} \quad H_3. \tag{43}$$

Using (41)–(43) and (19), we establish

$$\int_{H_2} \nu_i \, |D_i \, z|^{q_i} \, dx \leqslant 2^n (c_3 + 1) \, k \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx, \tag{44}$$

$$\int_{H_3} \nu_i \, |D_i \, z|^{q_i} \, dx \leqslant 2^n (c_3 + 1) \, k \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx. \tag{45}$$

Finally, Propositions 2 and 1 imply that

$$D_i z = 0 \quad \text{a.e. in} \quad H_4. \tag{46}$$

From (39), (40), and (44)–(46) we deduce

$$\int_\Omega \nu_i \, |D_i \, T_k(z)|^{q_i} \, dx \leqslant 2^n (c_3 + 1) \, k \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}.$$

From this fact and (37) it follows that for every $k \geqslant 1$ and $h \geqslant k + 1$,

$$\left(\int_{\{|u_1 - u_2| < k, |u_1 - v| < h, |u_2 - v| < h\}} |u_1 - u_2|^{n/(n-1)} dx \right)^{(n-1)/n} \leqslant$$

$$\leqslant c_0 \, c_6 \, k \sum_{i=1}^{n} \left\{ \int_{\{|u_1 - v| \geqslant h-k\}} \Phi_1 \, dx + \int_{\{|u_2 - v| \geqslant h-k\}} \Phi_2 \, dx \right\}^{1/q_i}.$$

The latter result and assertion (35) allow to conclude that for every $k \geqslant 1$,

$$\int_{\{|u_1 - u_2| < k\}} |u_1 - u_2|^{n/(n-1)} dx = 0.$$

Hence, $u_1 = u_2$ a.e. in Ω. □

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